

NON-ESCAPING SETS  
IN  
CONFORMAL DYNAMICAL SYSTEMS  
AND  
SINGULAR PERTURBATIONS  
OF  
PERRON-FROBENIUS OPERATORS

MARK POLLICOTT AND MARIUSZ URBAŃSKI

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ABSTRACT. The study of escape rates for a ball in a dynamical systems has been much studied. Understanding the asymptotic behavior of the escape rate as the radius of the ball tends to zero is an especially subtle problem. In the case of hyperbolic conformal systems this has been addressed by various authors [2], [8], [11] and these results apply in the case of real one dimensional expanding maps and conformal expanding repellers, particularly hyperbolic rational maps.

In this paper we consider a far more general realm of conformal maps where the analysis is correspondingly more complicated. We prove the existence of escape rates and calculate them in the context of countable alphabets, either finite or infinite, uniformly contracting conformal graph directed Markov systems (see [14], [13]) with their special case of conformal countable alphabet iterated function systems. The reference measures are the projections of Gibbs/equilibrium states of Hölder continuous summable potentials from a countable alphabet subshifts of finite type to the limit set of the graph directed Markov system under consideration.

This goal is achieved firstly by developing the appropriate theory of singular perturbations of Perron-Frobenius (transfer) operators associated with countable alphabet subshifts of finite type and Hölder continuous summable potentials, see [12] and [14] for the theory of such unperturbed operators, and [11] and [8] for singular perturbations which motivated our methods.

In particular, this includes, as a second ingredient in its own right, the asymptotic behavior of leading eigenvalues of perturbed operators and their first and second derivatives.

Our third ingredient is to relate the geometry and dynamics, roughly speaking to relate the case of avoiding cylinder sets and that of avoiding Euclidean geometric balls. Towards this end, in particular, we investigate in detail thin boundary properties relating the measures of thin annuli to the measures of the balls they enclose. In particular we clarify the results in the case of expanding repellers and conformal graph directed Markov systems with finite alphabet.

The setting of conformal graph directed Markov systems is interesting in its own and moreover, in our approach, it forms the key ingredient for further results about other conformal systems. These include topological Collet-Eckmann multimodal interval maps and rational maps of the Riemann sphere (an equivalent formulation is to be uniformly hyperbolic on periodic points), and also a large class of transcendental meromorphic functions, such as those introduced and explored in [16] and [17].

Our approach here is firstly to note that all of these systems yield some sets, commonly referred to as nice ones, the first return (induced) map to which is isomorphic to a conformal countable alphabet iterated function system with some additional properties. Secondly, with the help of appropriate large deviation results, to relate escape rates of the original system with the induced one and then to apply the results of graph directed Markov systems. The reference measures are again Gibbs/equilibrium states of some large classes of Hölder continuous potentials.

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## 1. INTRODUCTION

The escape rate for a dynamical system is a natural concept which describes the speed at which orbits of points first enter a small region of the space. The size of these sets is usually measured with respect to an appropriate probability. More precisely, given a metric space  $(X, d)$ , we can consider a (usually) continuous transformation  $T : X \rightarrow X$  and a ball

$$B(z, \epsilon) = \{x \in X : d(x, z) < \epsilon\}$$

of radius  $\epsilon > 0$  about a given point  $z$ . We then obtain an open system by removing  $B(z, \epsilon)$  and considering the new space  $X \setminus B(z, \epsilon)$  and truncating those orbits that land in the ball  $B(z, \epsilon)$ , which can be thought of informally as a “hole” in the system. This is the reason that many authors speak of *escape rates* for the system, whereas it might be a more suitable nomenclature to call them *avoidable sets*.

We can then consider for each  $n > 0$  the set  $R_n(z, \epsilon)$  of points  $x \in X$  for which all the first  $n$  terms in the orbit omit the ball, i.e.,  $x, T(x), \dots, T^{n-1}(x) \notin B(z, \epsilon)$ . It is evident that these sets are nested in both parameters  $\epsilon$  and  $n$ , i.e.,

$$R_{n+1}(z, \epsilon) \subset R_n(z, \epsilon)$$

for all  $n \geq 1$  and that

$$R_n(z, \epsilon) \subset R_n(z, \epsilon')$$

for  $\epsilon > \epsilon'$ . We can first ask about the behavior of the size of the sets  $R_n(z, \epsilon)$  as  $n \rightarrow +\infty$ .

If we assume that  $\mu$  is a  $T$ -invariant probability measure, say, then we can consider the measures  $\mu(R_n(z, \epsilon))$  of the sets  $R_n(z, \epsilon)$  as  $n \rightarrow +\infty$ . In particular, we can define the *lower and upper escape rates* respectively as

$$\underline{R}_\mu(B(z, \epsilon)) = - \overline{\lim}_{n \rightarrow +\infty} \frac{1}{n} \log \mu(R_n(z, \epsilon)) \quad \text{and} \quad \overline{R}_\mu(B(z, \epsilon)) = - \underline{\lim}_{n \rightarrow +\infty} \frac{1}{n} \log \mu(R_n(z, \epsilon)).$$

One can further consider how the escape rate behaves as the radius of the ball  $\epsilon$  tends to zero. An early influential result in this direction was [32]. Perhaps the simplest case is that of the doubling map  $E_2 : [0, 1) \rightarrow [0, 1)$  defined by  $E_2(x) = 2x \pmod{1}$  and the usual Lebesgue measure  $\lambda$ . For this example it was Bunimovitch and Yurchenko [2] (see also [11]) who showed the following, perhaps surprising, result showing that

$$(1.1) \quad \lim_{\epsilon \rightarrow 0} \frac{\underline{R}_\lambda(B(z, \epsilon))}{\lambda(B(z, \epsilon))} = \lim_{\epsilon \rightarrow 0} \frac{\overline{R}_\lambda(B(z, \epsilon))}{\lambda(B(z, \epsilon))} = \begin{cases} 1 & \text{if } z \text{ is not periodic} \\ 1 - 2^{-p} & \text{if } E_2^p(z) = z \text{ is periodic (with minimal period } p). \end{cases}$$

In particular, the asymptotic escape rate can only take a certain set of values which are determined by the periods of periodic points. More results in this direction followed, particularly in [11] and [8]. We will return to generalizations of these ideas after discussing a related problem.

One can also ask what is happening to full escaping/avoiding sets when  $\epsilon > 0$  decreases to zero. By full escaping sets we mean the sets of the form

$$K_z(\epsilon) = \{x \in X : T^n(x) \notin B(z, \epsilon) \forall n \geq 0\}$$

Such sets are usually of measure  $\mu$  zero, but there is another natural quantity to measure their size and complexity, namely their Hausdorff dimension. The second named author already addressed this question in the early 80s by showing in [30] and [31] that in the case of the doubling map  $E_2$ , or more generally, of any map  $E_q(x) = qx \pmod{1}$ ,  $q$  being

an integer greater than 1 in absolute value, or even more generally, in the case of any  $C^{1+\eta}$  expanding map of the unit circle, the map

$$\varepsilon \mapsto \text{HD}(K_z(\varepsilon))$$

is continuous. Moreover, it was also shown that this function is almost everywhere locally constant, in fact, the set of points where it fails to be locally constant is a closed set of Hausdorff dimension 1 and Lebesgue measure zero. Rather curiously, the local Hausdorff dimension at each point  $r$  of this set is equal to  $\text{HD}(K_z(\varepsilon))$ . All of this suggests that it is interesting to study the asymptotic properties of  $\text{HD}(K_z(\varepsilon))$  when  $\varepsilon \searrow 0$ . Andrew Ferguson and the first named author of this paper took up the challenge by proving in [8] that

$$(1.2) \quad \lim_{\varepsilon \rightarrow 0} \frac{\text{HD}(J) - \text{HD}(K_z(r))}{\mu_b(B(z, r))} = \begin{cases} 1/\chi_{\mu_b} & \text{if } z \text{ is not a periodic point of } T \\ \frac{1 - |(T^p)'(z)|^{-1}}{\chi_{\mu_b}} & \text{if } z \text{ is a periodic point of } T \text{ with prime period } p \geq 1. \end{cases}$$

in the case of any conformal expanding repeller  $T : J \rightarrow J$ ; where  $b$  here is just the Hausdorff dimension  $\text{HD}(J)$  and  $\mu_b$  is the equilibrium state of the Hölder continuous potential  $J \ni x \mapsto -b \log |T'(x)|$ . They have also established the analogue of (1.1) for such systems.

The approach of [8] was based on the method of singular perturbations of the Perron–Frobenius operators determined by the open sets  $B(z, \varepsilon)$ . They first did this for neighborhoods of  $z$  formed from finite unions of cylinders of  $n$ th refinements of a Markov partition and then used appropriate approximation. This required leaving the realm of the familiar Banach space of Hölder continuous functions, to work with a more refined space, and they applied the seminal results of Keller and Liverani from [11] to control the spectral properties of perturbed operators.

In the current paper we want to understand the escape rates, in the sense of equations (1.1) and (1.2), of essentially all conformal dynamical systems with an appropriate type of expanding dynamics. By this we primarily mean all topologically exact piecewise smooth maps of the interval  $[0, 1]$ , many rational functions of the Riemann sphere  $\widehat{\mathbb{C}}$  with degree  $\geq 2$ , a vast class of transcendental meromorphic functions from  $\mathbb{C}$  to  $\widehat{\mathbb{C}}$ , and last, but not least, the class of all countable alphabet conformal iterated function systems, and somewhat more generally, the class of all countable alphabet conformal graph directed Markov systems. This last class, i.e the collection of all countable alphabet conformal iterated function systems (IFSs), has a special status for us. The reasons for this are two-folded. Firstly, this class is interesting by itself, and secondly, by means of appropriate inducing schemes (involving the first return map), it is our indispensable tool for understanding the escape rates of all other systems mentioned above.

In order to deal with escape rates for countable alphabet conformal IFSs and conformal graph directed Markov systems (GDMSs), motivated by the work [8] of Andrew Ferguson and the first named author of this paper, we first develop the singular perturbation theory

for Perron-Frobenius operators associated to Hölder continuous summable potentials on countable alphabet shift of finite type symbol space. A comprehensive account of the thermodynamic formalism in the symbolic context can be found in [14], cf. also [12] and [13]. The general approach to control these perturbations is again based on the spectral results of Keller and Liverani from [11]. The perturbations in the case of a countable infinite alphabet require further refinement of the Banach space on which the original and perturbed Perron-Frobenius operators act. This space,  $\mathcal{B}_\theta$ , is defined already in the beginning of Section 3. Its definition, through the definition of the norm, involves the corresponding Gibbs/equilibrium measures. These measures play a further prominent role when investigating singular perturbations. Qualitatively new difficulties here, caused by an infinite alphabet, are many fold and a great deal of them are related to the facts that the symbol space  $E_A^\infty$  need not longer be compact, that there are infinitely many cylinders of given finite length, and that summable (particular geometric) potentials are unbounded in the supremum norm. Some remedy to this unboundedness issue is our repetitive use of Hölder inequalities rather than estimating by the supremum norms.

Having analyzed the symbolic part of the problem, we turn to escape rates for conformal GDMSs. With regard to formula (1.1), we consider the, already mentioned, measures on the limit set of the given conformal GDMS, that are projections of Gibbs/equilibrium states of Hölder continuous potentials from the symbol space. With respect to formula (1.2), we must consider geometric potentials, i.e. those of the form

$$E_A^\infty \ni \omega \mapsto t \log |\varphi'_{\omega_0}(\pi_{\mathcal{S}}(\sigma(\omega)))| \in \mathbb{R}$$

where  $\pi : E_A^\infty \rightarrow X$  is the canonical map for modelling the dynamics on  $X$ . Of particular interest are those for which  $t$  is close to  $b_{\mathcal{S}}$ , the Bowen parameter of the system conformal GDMS, which is defined as the only solution to the pressure equation

$$P(\sigma, t \log |\varphi'_{\omega_0}(\pi_{\mathcal{S}}(\sigma(\omega)))|) = 0,$$

provided that such solution exists. We can then consider the projection of the Gibbs/equilibrium state  $\mu_b$  for the potential  $t \log |\varphi'_{\omega_0}(\pi_{\mathcal{S}}(\sigma(\omega)))|$  on the limit set  $J_{\mathcal{S}}$ . This leads to the particularly technically involved task of calculating the asymptotic behavior of derivatives  $\lambda'_n(t)$  and  $\lambda''_n(t)$  of leading eigenvalues of perturbed operators when the integer  $n \geq 0$  diverges to infinity and the parameter  $t$  approaches  $b_{\mathcal{S}}$ . This is again partially due to unboundedness of the function  $E_A^\infty \ni \omega \mapsto t \log |\varphi'_{\omega_0}(\pi_{\mathcal{S}}(\sigma(\omega)))| \in \mathbb{R}$  in the supremum norm and partially due to lack of uniform topological mixing on the sets  $K_z(\varepsilon)$ .

We say that a set  $J \subseteq \mathbb{R}^d$ ,  $d \geq 1$ , is geometrically irreducible if it is not contained in any countable union of conformal images of hyperplanes or spheres of dimension  $\leq d - 1$  (see Definition 9.4). Our most general results about escape rates for conformal GDMSs can now be formulated in the following four theorems. We postpone detailed definitions of the hypotheses until later.

**Theorem 1.1.** *Let  $\mathcal{S} = \{\varphi_e\}_{e \in E}$  be a finitely primitive Conformal GDMS with limit set  $J_{\mathcal{S}}$ . Let  $\varphi : E_A^\infty \rightarrow \mathbb{R}$  be a Hölder continuous summable potential with equilibrium/Gibbs*

state  $\mu_\varphi$ . Assume that the measure  $\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}$  is weakly boundary thein (WBT) at a point  $z \in J_{\mathcal{S}}$ . If  $z$  is either

- (a) not pseudo-periodic,
- or

- (b) uniquely periodic, it belongs to  $\text{Int}X$  (and  $z = \pi(\xi^\infty)$  for a (unique) irreducible word  $\xi \in E_A^*$ ), and  $\varphi$  is the amalgamated function of a summable Hölder continuous system of functions,

then, with  $\underline{R}_{\mathcal{S},\varphi}(B(z, \varepsilon)) := \underline{R}_{\mu_\varphi}(\pi_{\mathcal{S}}^{-1}(B(z, \varepsilon)))$  and  $\overline{R}_{\mathcal{S},\varphi}(B(z, \varepsilon)) := \overline{R}_{\mu_\varphi}(\pi_{\mathcal{S}}^{-1}(B(z, \varepsilon)))$ , we have that

$$(1.3) \quad \lim_{\varepsilon \rightarrow 0} \frac{\underline{R}_{\mathcal{S},\varphi}(B(z, \varepsilon))}{\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}(B(z, \varepsilon))} = \lim_{\varepsilon \rightarrow 0} \frac{\overline{R}_{\mathcal{S},\varphi}(B(z, \varepsilon))}{\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}(B(z, \varepsilon))} =$$

$$= d_\varphi(z) := \begin{cases} 1 & \text{if (a) holds} \\ 1 - \exp(S_p \varphi(\xi) - pP(\varphi)) & \text{if (b) holds,} \end{cases}$$

where in (b),  $\{\xi\} = \pi_{\mathcal{S}}^{-1}(z)$  and  $p \geq 1$  is the prime period of  $\xi$  under the shift map.

**Theorem 1.2.** Assume that  $\mathcal{S}$  is a finitely primitive conformal GDMS whose limit set  $J_{\mathcal{S}}$  is geometrically irreducible. Let  $\varphi : E_A^\infty \rightarrow \mathbb{R}$  be a Hölder continuous strongly summable potential. As usual, denote its equilibrium/Gibbs state by  $\mu_\varphi$ . Then, with  $R_{\mathcal{S},\varphi}(B(z, \varepsilon)) := R_{\mu_\varphi}(\pi_{\mathcal{S}}^{-1}(B(z, \varepsilon)))$ , we have that

$$(1.4) \quad \lim_{\varepsilon \rightarrow 0} \frac{R_{\mathcal{S},\varphi}(B(z, \varepsilon))}{\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}(B(z, \varepsilon))} = \lim_{\varepsilon \rightarrow 0} \frac{\overline{R}_{\mathcal{S},\varphi}(B(z, \varepsilon))}{\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}(B(z, \varepsilon))} = 1$$

for  $\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}$ -a.e. point  $z$  of  $J_{\mathcal{S}}$ .

These two theorems address the issue of (1.1). We would like to bring to the reader's attention a preprint [1] by H. Bruin, M.F. Demers and M. Todd, with results related to the above, which we recently received. In regard to (1.2), we have proved for conformal GDMSs the following two theorems. In regard to (1.2), we have proved the following two theorems for conformal GDMSs.

**Theorem 1.3.** Let  $\mathcal{S}$  be a finitely primitive strongly regular conformal GDMS. Assume both that  $\mathcal{S}$  is (WBT) and the parameter  $b_{\mathcal{S}}$  is powering at some point  $z \in J_{\mathcal{S}}$  which is either

- (a) not pseudo-periodic or else
- (b) uniquely periodic and belongs to  $\text{Int}X$  (and  $z = \pi(\xi^\infty)$  for a (unique) irreducible word  $\xi \in E_A^*$ ).

Then

$$(1.5) \quad \lim_{r \rightarrow 0} \frac{\text{HD}(J_{\mathcal{S}}) - \text{HD}(K_z(r))}{\mu_b(\pi^{-1}(B(z, r)))} = \begin{cases} 1/\chi_{\mu_b} & \text{if (a) holds} \\ (1 - |\varphi'_\xi(z)|)/\chi_{\mu_b} & \text{if (b) holds.} \end{cases}$$

**Corollary 1.4.** *If  $\mathcal{S}$  be a finitely primitive strongly regular conformal GDMS whose limit set  $J_{\mathcal{S}}$  is a geometrically irreducible, then*

$$(1.6) \quad \lim_{r \rightarrow 0} \frac{\text{HD}(J_{\mathcal{S}}) - \text{HD}(K_z(r))}{\mu_b(\pi^{-1}(B(z, r)))} = \frac{1}{\chi_{\mu_b}}$$

at  $\mu_{b_{\mathcal{S}}} \circ \pi^{-1}$ -a.e. point  $z$  of  $J_{\mathcal{S}}$ .

As we have previously remarked, these four results are of independent interest, but they also provide a gateway to all other results on escape rates in this paper. There are necessarily several technical terms involved in formulations of these theorems. However, we hope that they do not obscure the overall meaning of the four theorems and all terms are carefully introduced and explained in appropriate sections dealing with them.

We would however like to comment on one of these terms, namely on (WBT). Its meaning can be understood as follows. Let

$$A(z; r, R) := B(z, R) \setminus \overline{B(z, r)}$$

be the annulus centered at  $z$  with the inner radius  $r$  and the outer radius  $R$ . We say that a finite Borel measure  $\mu$  is weakly boundary thin (WBT) (with exponent  $\beta > 0$ ) at the point  $x$  if

$$\lim_{r \rightarrow 0} \frac{\mu(A_{\mu}^{\beta}(x, r))}{\mu(B(x, r))} = 0,$$

where we denote

$$A_{\mu}^{\beta}(x, r) := A(x; r - \mu(B(x, r))^{\beta}, r + \mu(B(x, r))^{\beta}).$$

This is a version of the problem of thin annuli, one that is notoriously challenging in dealing with the issue of relating dynamical and geometric properties, and which is particularly acute in the contexts of escape rates and return rates. Due to the breakthrough of [19], where some strong versions of the thin annuli properties are proved, we have been able in the current paper to prove (WBT) for almost all points, which is reflected in both Theorem 10.11 and Corollary 12.3.

In the case of finite alphabets  $E$  we have the following two results.

**Theorem 1.5.** *Let  $\mathcal{S} = \{\varphi_e\}_{e \in E}$  be a primitive conformal GDMS with a finite alphabet  $E$  acting in the space  $\mathbb{R}^d$ ,  $d \geq 1$ . Assume that either  $d = 1$  or that the system  $\mathcal{S}$  is geometrically irreducible. Let  $\varphi : E_A^{\infty} \rightarrow \mathbb{R}$  be a Hölder continuous potential. As usual, denote its equilibrium/Gibbs state by  $\mu_{\varphi}$ . Let  $z \in J_{\mathcal{S}}$  be arbitrary. If either  $z$  is*

(a) *not pseudo-periodic,*

*or*

(b) *uniquely periodic, it belongs to  $\text{Int}X$  (and  $z = \pi(\xi^{\infty})$  for a (unique) irreducible word  $\xi \in E_A^*$ ), and  $\varphi$  is the amalgamated function of a summable Hölder continuous system of functions,*



then,

$$\begin{aligned}
 (1.7) \quad \lim_{\varepsilon \rightarrow 0} \frac{\underline{R}_{\mathcal{S},\varphi}(B(z, \varepsilon))}{\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}(B(z, \varepsilon))} &= \lim_{\varepsilon \rightarrow 0} \frac{\overline{R}_{\mathcal{S},\varphi}(B(z, \varepsilon))}{\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}(B(z, \varepsilon))} = \\
 &= d_\varphi(z) := \begin{cases} 1 & \text{if (a) holds} \\ 1 - \exp(S_p \varphi(\xi) - pP(\varphi)) & \text{if (b) holds,} \end{cases}
 \end{aligned}$$

where in (b),  $\{\xi\} = \pi_{\mathcal{S}}^{-1}(z)$  and  $p \geq 1$  is the prime period of  $\xi$  under the shift map.

**Theorem 1.6.** *Let  $\mathcal{S} = \{\varphi_e\}_{e \in E}$  be a primitive conformal GDMS with a finite alphabet  $E$  acting in the space  $\mathbb{R}^d$ ,  $d \geq 1$ . Assume that either  $d = 1$  or that the system  $\mathcal{S}$  is geometrically irreducible. Let  $z \in J_{\mathcal{S}}$  be arbitrary. If either  $z$  is*

- (a) *not pseudo-periodic or else*
- (b) *uniquely periodic and belongs to  $\text{Int}X$  (and  $z = \pi(\xi^\infty)$  for a (unique) irreducible word  $\xi \in E_A^*$ ).*

Then

$$(1.8) \quad \lim_{r \rightarrow 0} \frac{\text{HD}(J_{\mathcal{S}}) - \text{HD}(K_z(r))}{\mu_b(\pi^{-1}(B(z, r)))} = \begin{cases} 1/\chi_{\mu_b} & \text{if (a) holds} \\ (1 - |\varphi'_\xi(z)|)/\chi_{\mu_b} & \text{if (b) holds.} \end{cases}$$

For these two theorems the two Thin Annuli Properties, Theorem 9.9 and Theorem 9.10, were also instrumental. With having both Theorem 10.12 and Theorem 12.5 proved we have fully recovered the results of [8].

As we have already explained, our next goal in this paper is to get the existence of escape rates in the sense of (1.1) and (1.2) for all topologically exact piecewise smooth maps of the interval  $[0, 1]$ , many rational functions of the Riemann sphere  $\widehat{\mathbb{C}}$  with degree  $\geq 2$ , and a vast class of transcendental meromorphic functions from  $\mathbb{C}$  to  $\widehat{\mathbb{C}}$ . In order to do this we employ two principle tools. The first is formed by the escape rates results, described above in detail, for the class of all countable alphabet conformal graph directed Markov systems. The second is a method based on the first return (induced) map developed in Section 14, Section 15, and Section 16. This method closely relates the escape rates of the original map and the induced map. It turns out that for the above mentioned classes of systems one can find a set of positive measure which gives rise to a first return map which is isomorphic to a countable alphabet conformal IFS or full shift map; the task being highly non-trivial and technically involved. But this allows us to conclude, for suitable systems, the existence of escape rates in the sense of (1.1) and (1.2). However, in order to reach this conclusion we need to know some non-trivial properties of the original systems. Firstly, that the tails of the first return time and the first entrance time decay exponentially fast, and secondly that the Large Deviation Property (LDP) of Section 15 holds. This in turn leads to Theorem 16.6, a kind of Large Deviation Theorem.

We shall now describe in some detail the above mentioned applications to (quite) general conformal systems. We start with one-dimensional systems. We consider the class of

topologically exact piecewise  $C^3$ -smooth multimodal maps  $T$  of the interval  $I = [0, 1]$  with non-flat critical points and uniformly expanding periodic points, the property commonly referred to as Topological Collet–Eckmann. Topological exactness means that for every non-empty subset  $U$  of  $I$  there exists an integer  $n \geq 0$  such that  $T^n(U) = I$ . Furthermore, our multimodal map  $T : I \rightarrow I$  is assumed to be tame, meaning that

$$\overline{\text{PC}(T)} \neq I,$$

where

$$\text{Crit}(T) := \{c \in I : T'(c) = 0\}$$

is the critical set for  $T$  and

$$\text{PC}(T) := \bigcup_{n=1}^{\infty} T^n(\text{Crit}(T)),$$

is the postcritical set of  $T$ . A familiar example would be the famous unimodal map  $x \mapsto \lambda x(1-x)$  with  $0 < \lambda < 4$  for which the critical point  $1/2$  is not in its own omega limit set, for example where  $\lambda$  is a Misiurewicz point.

The class of potentials, called acceptable in the sequel, is provided by all Lipschitz continuous functions  $\psi : I \rightarrow \mathbb{R}$  for which

$$\sup(\psi) - \inf(\psi) < h_{\text{top}}(T).$$

The first escape rates theorem in this setting is this.

**Theorem 1.7.** *Let  $T : I \rightarrow I$  be a tame topologically exact Topological Collet–Eckmann map. Let  $\psi : I \rightarrow \mathbb{R}$  be an acceptable potential. Let  $z \in I \setminus \overline{\text{PC}(T)}$  be a recurrent point. Assume that the equilibrium state  $\mu_\psi$  is (WBT) at  $z$ . Then*

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{R_{\mu_\psi}(B(z, \varepsilon))}{\mu_\psi(B(z, \varepsilon))} &= \lim_{\varepsilon \rightarrow 0} \frac{\overline{R}_{\mu_\psi}(B(z, \varepsilon))}{\mu_\psi(B(z, \varepsilon))} = \\ &= \begin{cases} 1 & \text{if } z \text{ is not any periodic point of } T, \\ 1 - \exp(S_p \psi(z) - pP(f, \psi)) & \text{if } z \text{ is a periodic point of } T. \end{cases} \end{aligned}$$

We have used here the usual notation

$$S_p \psi(x) = \sum_{k=0}^{p-1} \psi(T^k(x))$$

of Birkhoff's sums, and  $P(f, \psi)$  denotes the topological pressure of the potential  $\psi$  with respect to the dynamical system  $T : I \rightarrow I$ . We have also the following.

**Theorem 1.8.** *Let  $T : I \rightarrow I$  be a tame topologically exact Topological Collet–Eckmann map. Let  $\psi : I \rightarrow \mathbb{R}$  be an acceptable potential. Then*

$$\lim_{\varepsilon \rightarrow 0} \frac{R_{\mu_\psi}(B(z, \varepsilon))}{\mu_\psi(B(z, \varepsilon))} = \lim_{\varepsilon \rightarrow 0} \frac{\overline{R}_{\mu_\psi}(B(z, \varepsilon))}{\mu_\psi(B(z, \varepsilon))} = 1$$

for  $\mu_\psi$ -a.e. point  $z \in I$ .

In order to address formula (1.2) in this context we need a stronger assumption on the map  $T : I \rightarrow I$ . Our multimodal map  $T : I \rightarrow I$  is said to be subexpanding if

$$\text{Crit}(T) \cap \overline{\text{PC}(T)} = \emptyset.$$

It is evident that each subexpanding map is tame and it is not hard to see that the subexpanding property entails being Topological Collet–Eckmann. It is well known that in this case there exists a unique Borel probability  $T$ -invariant measure  $\mu$  absolutely continuous with respect to Lebesgue measure  $\lambda$ . In fact,  $\mu$  is equivalent to  $\lambda$  and (therefore) has full topological support. It is ergodic, even  $K$ -mixing, has Rokhlin's natural extension metrically isomorphic to some two sided Bernoulli shift. The Radon–Nikodym derivative  $\frac{d\mu}{d\lambda}$  is uniformly bounded above and separated from zero on the complement of every fixed neighborhood of  $\overline{\text{PC}(T)}$ . We prove in this setting the following.

**Theorem 1.9.** *Let  $T : I \rightarrow I$  be a topologically exact multimodal subexpanding map. Fix  $\xi \in I \setminus \overline{\text{PC}(T)}$ . Assume that the parameter 1 is powering at  $\xi$  with respect to the conformal GDS  $\mathcal{S}_T$  defined in Section 17. Then the following limit exists, is finite, and positive:*

$$\lim_{r \rightarrow 0} \frac{1 - \text{HD}(K_\xi(r))}{\mu(B(\xi, r))}.$$

**Theorem 1.10.** *If  $T : I \rightarrow I$  is a topologically exact multimodal subexpanding map, then for Lebesgue–a.e. point  $\xi \in I \setminus \overline{\text{PC}(T)}$  the following limit exists, is finite and positive:*

$$\lim_{r \rightarrow 0} \frac{1 - \text{HD}(K_\xi(r))}{\mu(B(\xi, r))}.$$

We now turn to complex one-dimensional maps. Let  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  be a rational map of the Riemann sphere with degree  $\deg(f) \geq 2$ . The sets  $\text{Crit}(f)$  and  $\text{PC}(f)$  have the same meaning as for the multimodal maps of the interval  $I$ . Let  $\psi : \widehat{\mathbb{C}} \rightarrow \mathbb{R}$  be a Hölder continuous function. Following [6] we say that  $\psi : \widehat{\mathbb{C}} \rightarrow \mathbb{R}$  has a pressure gap if

$$(1.9) \quad nP(f, \psi) - \sup(\psi_n) > 0$$

for some integer  $n \geq 1$ . It was proved in [6] that there exists a unique equilibrium state  $\mu_\psi$  for such  $\psi$ . Some more ergodic properties of  $\mu_\psi$  were established there, and a fairly extensive account of them was provided in [29]. For example, if  $\psi = 0$  then  $P(f, 0) = \log \deg(f) > 0$  is the topological entropy of  $f$  and the condition automatically holds. More generally, it always holds whenever

$$\sup(\psi) - \inf(\psi) < h_{\text{top}}(f) (= \log \deg(f)).$$

We would like to also add that (1.9) always holds (with all  $n \geq 0$  sufficiently large) if the function  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  restricted to its Julia set is expanding (also frequently referred to as hyperbolic). This is the best understood and the easiest to deal with class of rational

functions. The rational map  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  is said to be expanding if the restriction  $f|_{J(f)} : J(f) \rightarrow J(f)$  satisfies

$$(1.10) \quad \inf\{|f'(z)| : z \in J(f)\} > 1$$

or, equivalently,

$$(1.11) \quad |f'(z)| > 1$$

for all  $z \in J(f)$ . Another, topological, characterization of expandingness is the following.

**Fact 1.11.** A rational function  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  is expanding if and only if

$$J(f) \cap \overline{\text{PC}(f)} = \emptyset.$$

It is immediate from this characterization that all the polynomials  $z \mapsto z^d$ ,  $d \geq 2$ , are expanding along with their small perturbations  $z \mapsto z^d + \varepsilon$ ; in fact expanding rational functions are commonly believed to form the vast majority amongst all rational functions.

Being a tame rational function and Topological Collet–Eckmann both mean the same as in the setting of multimodal interval maps. Nowadays this property is somewhat more frequently used in its equivalent form of exponential shrinking (see (18.3)) (ESP), and we thus follow tradition. All expanding functions are tame and (ESP). Finally, as in the context of interval maps, we have the following.

**Theorem 1.12.** *Let  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  be a tame rational function having the exponential shrinking property (ESP). Let  $\psi : \widehat{\mathbb{C}} \rightarrow \mathbb{R}$  be a Hölder continuous potential with pressure gap. Let  $z \in J(f) \setminus \overline{\text{PC}(f)}$  be recurrent. Assume that the equilibrium state  $\mu_\psi$  is (WBT) at  $z$ . Then*

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{R_{\mu_\psi}(B(z, \varepsilon))}{\mu_\psi(B(z, \varepsilon))} &= \lim_{\varepsilon \rightarrow 0} \frac{\overline{R}_{\mu_\psi}(B(z, \varepsilon))}{\mu_\psi(B(z, \varepsilon))} \\ &= \begin{cases} 1 & \text{if } z \text{ is not a periodic point for } f, \\ 1 - \exp(S_p \psi(z) - pP(f, \psi)) & \text{if } z \text{ is a periodic point of } f. \end{cases} \end{aligned}$$

**Corollary 1.13.** *Let  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  be a tame rational function having the exponential shrinking property (ESP) whose Julia set  $J(f)$  is geometrically irreducible. If  $\psi : \widehat{\mathbb{C}} \rightarrow \mathbb{R}$  is a Hölder continuous potential with pressure gap, then*

$$\lim_{\varepsilon \rightarrow 0} \frac{R_{\mu_\psi}(B(z, \varepsilon))}{\mu_\psi(B(z, \varepsilon))} = \lim_{\varepsilon \rightarrow 0} \frac{\overline{R}_{\mu_\psi}(B(z, \varepsilon))}{\mu_\psi(B(z, \varepsilon))} = 1$$

for  $\mu_\psi$ -a.e.  $z \in J(f)$ .

As for the case of maps of an interval, in order to establish formula (1.2) in this context we need a stronger assumption on the rational map  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ . Because the Julia set need not be equal to  $\widehat{\mathbb{C}}$  (and usually it is not) the definition of subexpanding rational functions is somewhat more involved, see Definition 18.13. It is evident that each subexpanding map

is tame and it is not hard to see that being subexpanding entails also being Topological Collet–Eckmann. All expanding functions are necessarily subexpanding.

**Theorem 1.14.** *Let  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  be a subexpanding rational function of degree  $d \geq 2$ . Fix  $\xi \in J(f) \setminus \overline{\text{PC}(f)}$ . Assume that the measure  $\mu_h$  is (WBT) at  $\xi$  and the parameter  $h := \text{HD}(J(f))$  is powering at  $\xi$  with respect to the conformal GDS  $\mathcal{S}_f$  defined in Section 18. Then the following limit exists, is finite and positive:*

$$\lim_{r \rightarrow 0} \frac{\text{HD}(J(f)) - \text{HD}(K_\xi(r))}{\mu_h(B(\xi, r))}.$$

**Theorem 1.15.** *If  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  be a subexpanding rational function of degree  $d \geq 2$  whose Julia set  $J(f)$  is geometrically irreducible, then for  $\mu_h$ -a.e. point  $\xi \in J(f) \setminus \overline{\text{PC}(f)}$  the following limit exists, is finite and positive:*

$$\lim_{r \rightarrow 0} \frac{\text{HD}(J(f)) - \text{HD}(K_\xi(r))}{\mu_h(B(\xi, r))}.$$

**Remark 1.16.** We would like to note that if the rational function  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  is expanding (or hyperbolic as such functions are frequently called), then it is subexpanding and each Hölder continuous potential has a pressure gap. In particular all four theorems above pertaining to rational functions hold for it.

In both theorems  $\mu_h$  is a unique (ergodic) Borel probability  $f$ -invariant measure on  $J(f)$  equivalent to  $m_h$ , a unique  $h$ -conformal measure  $m_h$  on  $J(f)$  for  $f$ . This was proved studied in [34], comp. also [33].

The last applications are in the realm of transcendental meromorphic functions. There is a large class of such systems, introduced in [16] and [17] for which it is possible to build (see these two papers) a fairly rich and complete account of thermodynamic formalism. Applying again our escape rates theorems for conformal graph directed Markov systems, one prove in this setting four main theorems which are analogous of those for the multimodal maps of an interval and rational functions of the Riemann sphere. These can be found with complete proofs in Section 19, the last section of our manuscript.

## Part 1. Singular Perturbations of Countable Alphabet Symbol Space Classical Perron–Frobenius Operators

### 2. THE CLASSICAL ORIGINAL PERRON-FROBENIUS OPERATOR, GIBBS AND EQUILIBRIUM STATES, THERMODYNAMIC FORMALISM; PRELIMINARIES

In this section we present some notation and basic results on Thermodynamic Formalism as developed in [14], see also [13] and [4]. It will be the base for our subsequent work.

Let  $E$  be a countable, either finite or infinite, set, called in the sequel the alphabet. Let  $A : E \times E \rightarrow \{0, 1\}$  an arbitrary matrix. For every integer  $n \geq 0$  let

$$E_A^n := \{\omega \in E^n : A_{\omega_j \omega_{j+1}} = 1 \ \forall 0 \leq j \leq n-1\},$$

denote the finite words of length  $n$ , let

$$E_A^\infty := \{\omega \in E^\mathbb{N} : A_{\omega_j \omega_{j+1}} = 1 \ \forall j \geq 0\},$$

denote the space of one-sided infinite sequences, and let

$$E^* := \bigcup_{n=0}^{\infty} E^n, \quad \text{and} \quad E_A^* := \bigcup_{n=0}^{\infty} E_A^n.$$

be set of all finite strings of words, the former being without restrictions and the latter being called  $A$ -admissible.

We call elements of  $E_A^*$  and  $E_A^\infty$   $A$ -admissible. The matrix  $A$  is called finitely primitive (or aperiodic) if there exist an integer  $p \geq 0$  and a finite set  $\Lambda \subseteq E^p$  such that for all  $i, j \in E$  there exists  $\omega \in \Lambda$  such that  $i\omega j \in E_A^*$ . Denote by  $\sigma : E_A^\infty \rightarrow E_A^\infty$  the shift map, i. e. the map uniquely defined by the property that

$$\sigma(\omega)_n := \omega_{n+1}$$

for every  $n \geq 0$ . Fixing  $\theta \in (0, 1)$  endow  $E_A^\infty$  with the standard metric

$$d_\theta(\omega, \tau) := \theta^{|\omega \wedge \tau|},$$

where for every  $g \in E^* \cup E^\mathbb{N}$ ,  $|\gamma|$  denotes the length of  $\gamma$ , i. e. the unique  $n \in \mathbb{N} \cup \{\infty\}$  such that  $\gamma \in E^n$ . Given  $0 \leq k \leq |\gamma|$ , we set

$$\gamma|_k := \gamma_1 \gamma_2 \dots \gamma_k.$$

We then also define

$$[\gamma] := \{\omega \in E_A^\infty : \omega|_n = \gamma\},$$

and call  $[\gamma]$  the (initial) cylinder generated by  $\gamma$ . Let  $\varphi : E_A^\infty \rightarrow \mathbb{R}$  be a Hölder continuous function, called in the sequel potential. We assume that  $\varphi$  is summable, meaning that

$$\sum_{e \in E} \exp(\sup(\varphi|_{[e]}) < +\infty.$$

It is well known (see [14] or [12]) that the following limit

$$P(\varphi) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\omega \in E_A^n} \exp(\sup(\varphi|_{[\omega]}))$$

exists. It is called the topological pressure of  $\varphi$ . It was proved in [12] (compare [14]) that there exists a unique shift-invariant Gibbs/equilibrium measure  $\mu_\varphi$  for the potential  $\varphi$ . The Gibbs property means that

$$C_\varphi^{-1} \leq \frac{\mu_\varphi([\omega|_n])}{\exp(\varphi_n(\omega) - P(\varphi)n)} \leq C_\varphi$$

with some constant  $C_\varphi \geq 1$  for every  $\omega \in E_A^\infty$  and every integer  $n \geq 1$ , where here and in the sequel

$$g_n(\omega) := \sum_{j=0}^{n-1} g \circ \sigma^j$$

for every function  $g : E_A^\infty \rightarrow \mathbb{C}$ . For the measure  $\mu_\varphi$  being an equilibrium state for the potential  $\varphi$  means that

$$h_{\mu_\varphi}(\sigma) + \int_{E_A^\infty} \varphi d\mu_\varphi = P(\varphi).$$

It has been proved in [14] that

$$h_\mu(\sigma) + \int_{E_A^\infty} \varphi d\mu < P(\varphi)$$

for any other Borel probability  $\sigma$ -invariant measure  $\mu$  such that  $\int \varphi d\mu > -\infty$ . For every bounded function  $g : E_A^\infty \rightarrow \mathbb{R}$  define  $\mathcal{L}_\varphi(g) : E_A^\infty \rightarrow \mathbb{R}$  as follows

$$\mathcal{L}_\varphi(g)(\omega) := \sum_{e \in E : A_{e\omega_0} = 1} g(e\omega) \exp(\varphi(e\omega)).$$

Then  $\mathcal{L}_\varphi(g)$  is bounded again, and we get by induction that

$$\mathcal{L}_\varphi^k(g)(\omega) := \sum_{\tau \in E_A^k : A_{\tau_{k-1}\omega_0} = 1} g(\tau\omega) \exp(\varphi_k(\tau\omega)).$$

Let  $C_b(A)$  be the Banach space of all complex-valued bounded continuous functions defined on  $E_A^\infty$  endowed with the supremum norm  $\|\cdot\|_\infty$ . Let  $H_\theta^b(A)$  be its vector subspace consisting of all Lipschitz continuous functions with respect to the metric  $d_\theta$ . Equipped with the norm

$$(2.1) \quad H_\theta(g) := \|g\|_\infty + v_\theta(g),$$

where  $v_\theta(g)$  is the least constant  $C \geq 0$  such that

$$(2.2) \quad |g(\omega) - g(\tau)| \leq C d_\theta(\omega, \tau),$$

whenever  $d_\theta(\omega, \tau) \leq \theta$  (i. e.  $\omega_0 = \tau_0$ ), the vector space  $H_\theta^b(A)$  becomes a Banach space. It is easy to see that the operator  $\mathcal{L}_\varphi$  preserves both Banach spaces  $C_b(A)$  (as we have observed some half-page ago) and  $H_\theta^b(A)$  and also acts continuously on each of them. The adjective “original” indicates that we do not deal with its perturbations while “classical” refers to standard Banach spaces  $C_b(A)$  and  $H_\theta^b(A)$ . The following theorem, describing fully the spectral properties of  $\mathcal{L}_\varphi$ , has been proved in [14] and [12].

**Theorem 2.1.** *If  $A : E \times E \rightarrow \{0, 1\}$  is finitely primitive and  $\varphi \in H_\theta^b(A)$ , then*

- (a) *The spectral radius of the operator  $\mathcal{L}_\varphi$  considered as acting either on  $C_b(A)$  or  $H_\theta^b(A)$  is in both cases equal to  $e^{P(\varphi)}$ .*
- (b) *In both cases of (a) the number  $e^{P(\varphi)}$  is a simple eigenvalue of  $\mathcal{L}_\varphi$  and there exists corresponding to it an everywhere positive eigenfunction  $\rho_\varphi \in H_\theta^b(A)$  such that  $\log \rho_\varphi$  is a bounded function.*

- (c) *The remainder of the spectrum of the operator  $\mathcal{L}_\varphi : H_\theta^b(A) \rightarrow H_\theta^b(A)$  is contained in a closed disk centered at 0 with radius strictly smaller than  $e^{P(\varphi)}$ . In particular, the operator  $\mathcal{L}_\varphi : H_\theta^b(A) \rightarrow H_\theta^b(A)$  is quasi-compact.*
- (d) *There exists a unique Borel probability measure  $m_\varphi$  on  $E_A^\infty$  such that*

$$\mathcal{L}_\varphi^* m_\varphi = e^{P(\varphi)} m_\varphi,$$

*where  $\mathcal{L}_\varphi^* : C_b^*(A) \rightarrow C_b^*(A)$ , is the operator dual to  $\mathcal{L}_\varphi$  acting on the space of all bounded linear functionals from  $C_b(A)$  to  $\mathbb{C}$ .*

- (e) *If  $\rho_\varphi : E_A^\infty \rightarrow (0, \infty)$  is normalized so that  $m_\varphi(\rho_\varphi) = 1$ , then  $\rho_\varphi m_\varphi = \mu_\varphi$ , where, we recall, the latter is the unique shift-invariant Gibbs/equilibrium measure for the potential  $\varphi$ .*
- (e) *The Riesz projector  $Q_1 : H_\theta^b(A) \rightarrow H_\theta^b(A)$ , corresponding to the eigenvalue  $e^{P(\varphi)}$ , is given by the formula*

$$Q_1(g) = e^{P(\varphi)} m_\varphi(g) \rho_\varphi.$$

If we multiply the operator  $\mathcal{L}_\varphi : H_\theta^b(A) \rightarrow H_\theta^b(A)$  by  $e^{-P(\varphi)}$  and conjugate it via the linear homeomorphism

$$g \mapsto \rho_\varphi^{-1} g,$$

then the resulting operator  $T : H_\theta^b(A) \rightarrow H_\theta^b(A)$  has the same properties, described above, as the operator  $\mathcal{L}_\varphi$ , with  $e^{P(\varphi)}$  replaced by 1,  $\rho_\varphi$  by the function  $\mathbb{1}$  which is identically equal to 1, and  $m_\varphi$  replaced by  $\mu_\varphi$ . Since in addition it is equal to  $\mathcal{L}_{\tilde{\varphi}} : H_\theta^b(A) \rightarrow H_\theta^b(A)$  with

$$\tilde{\varphi} := \varphi - P(\varphi) + \log \rho_\varphi,$$

we will frequently deal with the operator  $\mathcal{L}_{\tilde{\varphi}}$  instead of  $\mathcal{L}_\varphi$ , exploiting its useful property

$$\mathcal{L}_{\tilde{\varphi}} \mathbb{1} = \mathbb{1}.$$

We will occasionally refer to  $\mathcal{L}_{\tilde{\varphi}}$  as fully normalized. Sometimes, we will only need the semi-normalized operator  $\mathcal{L}_\varphi$  given by the formula

$$\hat{\mathcal{L}}_\varphi := e^{-P(\varphi)} \mathcal{L}_\varphi.$$

It essentially differs from only by having  $e^{P(\varphi)}$  replaced by 1. Now we bring up two standard well-known technical facts about the above concepts. These can be found for example in [14].

**Lemma 2.2.** *There exists a constant  $M_\varphi \in (0, +\infty)$  such that*

$$|\varphi_k(\omega) - \varphi_k(\tau)| \leq M_\varphi \theta^m$$

*for all integers  $k, m \geq 1$ , and all words  $\omega, \tau \in E_A^\infty$  such that  $\omega|_{k+m} = \tau|_{k+m}$ .*

**Lemma 2.3.** *With the hypotheses of Lemma 2.2 and increasing the constant  $M_\varphi$  if necessary, we have that*

$$|1 - \exp(\varphi_k(\gamma\omega) - \varphi_k(\gamma\tau))| \leq M_\varphi |\varphi_k(\omega) - \varphi_k(\tau)|.$$



### 3. NON-STANDARD ORIGINAL PERRON-FROBENIUS OPERATOR; DEFINITION AND FIRST TECHNICAL RESULTS

We keep the setting of the previous section. We still deal with the original operator  $\mathcal{L}_\varphi$  but we let it act on a different non-standard Banach space  $\mathcal{B}_\theta$  defined below. This space is more suitable for consideration of perturbations of  $\mathcal{L}_\varphi$ .

Given a function  $g \in L^1(\mu_\varphi)$  and an integer  $m \geq 0$ , we define the function  $\text{osc}_m(g) : E_A^\infty \rightarrow [0, \infty)$  by the following formula:

$$(3.1) \quad \text{osc}_m(g)(\omega) := \text{ess sup} \{ |g(\alpha) - g(\beta)| : \alpha, \beta \in [\omega]_m \}$$

and

$$\text{osc}_0(g) := \text{ess sup}(g) - \text{ess inf}(g).$$

We further define:

$$(3.2) \quad |g|_\theta := \sup_{m \geq 0} \{ \theta^{-m} \|\text{osc}_m(g)\|_1 \},$$

where  $|\cdot|$  denotes the  $L^1$ -norm with respect to the measure  $\mu_\varphi$ . Note the subtle difference between this definition and the analogous one, which motivated us, from [8]. Therein in the analogue of formula (3.2) the supremum is taken over integers  $m \geq 1$  only. Including  $m = 0$  causes some technical difficulties, particularly the (tedious) part of the proof of Lemma 3.2 for the integer  $m = 0$ . However, without the case  $m = 0$  we would not be able to prove Lemma 3.1, in contrast to the finite alphabet case of [8], which is indispensable for our entire approach. The, previously announced, non-standard (it even depends on the dynamics – via  $\mu_\varphi$ ) Banach space is defined as follows:

$$\mathcal{B}_\theta := \{g \in L^1(\mu_\varphi) : |g|_\theta < +\infty\}$$

and we denote

$$(3.3) \quad \|g\|_\theta := \|g\|_1 + |g|_\theta.$$

Of course  $\mathcal{B}_\theta$  is a vector space and the function

$$(3.4) \quad \mathcal{B}_\theta \ni g \mapsto \|g\|_\theta$$

is a norm on  $\mathcal{B}_\theta$ . This is the non-standard Banach space we will be working with throughout the whole manuscript. We shall prove the following.

**Lemma 3.1.** *If  $g \in \mathcal{B}_\theta$ , then  $g$  is essentially bounded and*

$$\|g\|_\infty \leq \|g\|_\theta.$$

*Proof.* For all  $\omega \in E_A^\infty$ , we have

$$\begin{aligned} |g(\omega)| &\leq \left| \int_{E_A^\infty} g \, d\mu_\varphi + \text{osc}_0(g)(\omega) \right| = \left| \int_{E_A^\infty} g \, d\mu_\varphi + \int_{E_A^\infty} \text{osc}_0(g) \, d\mu_\varphi \right| \\ &\leq \int_{E_A^\infty} |g| \, d\mu_\varphi + \|\text{osc}_0(g)\|_1 \\ &\leq \|g\|_\theta. \end{aligned}$$

The proof is complete.  $\square$

From now on, unless otherwise stated, we assume that the potential  $\varphi : E_A^\infty \rightarrow \mathbb{R}$  is normalized (by adding a constant and a coboundary) so that

$$\mathcal{L}_\varphi \mathbb{1} = \mathbb{1}.$$

For ease of notation we also abbreviate  $\mathcal{L}_\varphi$  to  $\mathcal{L}$ . We shall prove the following.

**Lemma 3.2.** *There exists a constant  $C > 0$  for every integer  $k \geq 0$  and every  $g \in \mathcal{B}_\theta$ , we have*

$$|\mathcal{L}^k g|_\theta \leq C(\theta^k |g|_\theta + \|g\|_1).$$

*Proof.* For every  $e \in E$  let

$$E_A^k(e) := \{\gamma \in E_A^k : A_{\gamma_k e} = 1\}.$$

Fix first an integer  $m \geq 1$  and then  $\omega, \tau \in E_A^\infty$  such that  $\omega|_m = \tau|_m$ . Using Lemmas 2.2 and 2.3, we then get

$$\begin{aligned} |\mathcal{L}^k g(\omega) - \mathcal{L}^k g(\tau)| &\leq \sum_{\gamma \in E_A^k(\omega_1)} e^{\varphi_k(\gamma\omega)} |g(\gamma\omega) - g(\gamma\tau)| + \sum_{\gamma \in E_A^k(\omega_1)} |g(\gamma\tau)| |e^{\varphi_k(\gamma\omega)} - e^{\varphi_k(\gamma\tau)}| \\ &\leq \sum_{\gamma \in E_A^k(\omega_1)} \text{osc}_{k+m}(g)(\gamma\omega) e^{\varphi_k(\gamma\omega)} + \\ &\quad + \sum_{\gamma \in E_A^k(\omega_1)} |g(\gamma\tau)| e^{\varphi_k(\gamma\tau)} |1 - \exp(\varphi_k(\gamma\omega) - \varphi_k(\gamma\tau))| \\ &\leq \sum_{\gamma \in E_A^k(\omega_1)} \text{osc}_{k+m}(g)(\gamma\omega) e^{\varphi_k(\gamma\omega)} + \\ &\quad + \sum_{\gamma \in E_A^k(\omega_1)} |g(\gamma\tau)| e^{\varphi_k(\gamma\tau)} M_\varphi |\varphi_k(\gamma\omega) - \varphi_k(\gamma\tau)| \\ &\leq \mathcal{L}^k(\text{osc}_{k+m}(g))(\omega) + M_\varphi^2 \theta^m \sum_{\gamma \in E_A^k(\omega_1)} (|g(\gamma\omega)| + \text{osc}_{k+m}(g)(\gamma\omega)) e^{\varphi_k(\gamma\omega)} \\ &\leq \mathcal{L}^k(\text{osc}_{k+m}(g))(\omega) + M_\varphi^2 \theta^m \mathcal{L}^k(|g|)(\omega) + M_\varphi^2 \theta^m \mathcal{L}^k(\text{osc}_{k+m}(g))(\omega) \\ &\leq (1 + M_\varphi^2) \mathcal{L}^k(\text{osc}_{k+m}(g))(\omega) + M_\varphi^2 \theta^m \mathcal{L}^k(|g|)(\omega) \end{aligned}$$

Hence,

$$\text{osc}_m(\mathcal{L}^k g)(\omega) \leq (1 + M_\varphi^2) \mathcal{L}^k(\text{osc}_{k+m}(g))(\omega) + M_\varphi^2 \theta^m \mathcal{L}^k(|g|)(\omega)$$

Integrating against the measure  $\mu_\varphi$ , this yields

$$\begin{aligned}
 \theta^{-m} \|\text{osc}_m(\mathcal{L}^k g)\|_1 &\leq (1 + M_\varphi^2) \theta^{-m} \int_{E_A^\infty} \mathcal{L}^k(\text{osc}_{k+m}(g)) d\mu_\varphi + M_\varphi^2 \int_{E_A^\infty} \mathcal{L}^k(|g|) d\mu_\varphi \\
 (3.5) \quad &= (1 + M_\varphi^2) \theta^{-m} \int_{E_A^\infty} \text{osc}_{k+m}(g) d\mu_\varphi + M_\varphi^2 \int_{E_A^\infty} |g| d\mu_\varphi. \\
 &\leq (1 + M_\varphi^2) \theta^k |g|_\theta + M_\varphi^2 \|g\|_1 \\
 &\leq (1 + M_\varphi^2) (\theta^k |g|_\theta + \|g\|_1).
 \end{aligned}$$

Some separate considerations are needed if  $m = 0$ . However, we note that it would require no special treatment in the case of a full shift, i. e. when the incidence matrix  $A$  consists of 1s only. Let  $p \geq 1$  be the value in the definition of finite primitivity of the matrix  $A$ . Replacing  $p$  by a sufficiently large integral multiple, we will have that the set

$$E_A^p(a, b) := \{\alpha \in E_A^p : a\alpha b \in E_A^*\}$$

consisting of words of length  $p$  prefixed by  $a$  and suffixed by  $b$  is non-empty for all  $a, b \in E$  and it is countable infinite if the alphabet  $E$  is infinite. For every function  $h : E_A^\infty \rightarrow \mathbb{R}$  and every finite word  $\gamma \in E_A^*$  with associated cylinder  $[\gamma]$  consisting of all infinite sequences beginning with  $\gamma$  let  $\hat{h}(\gamma) \in \mathbb{R}$  be a number with the following two properties:

- (a)  $\hat{h}(\gamma) \in \overline{h([\gamma])}$  and
- (b)  $|\hat{h}(\gamma)| = \inf\{|h(\rho)| : \rho \in [\gamma]\}$ .

Let us introduce the following two functions:

$$\Delta_1 \mathcal{L}^{k+p}(g)(\rho) := \sum_{|\gamma|=k} \sum_{\alpha \in E_A^p(\gamma_k, \rho_1)} (g(\gamma\alpha\rho) e^{\varphi_k(\gamma\alpha\rho)} e^{\varphi_p(\alpha\rho)} - \hat{g}(\gamma) e^{\hat{\varphi}_k(\gamma)} e^{\varphi_p(\alpha\rho)})$$

and

$$\Delta_2 \mathcal{L}^{k+p}(g)(\omega, \tau) := \sum_{|\gamma|=k} \hat{g}(\gamma) e^{\hat{\varphi}_k(\gamma)} \left( \sum_{\alpha \in E_A^p(\gamma_k, \omega_1)} e^{\varphi_p(\alpha\omega)} - \sum_{\beta \in E_A^p(\gamma_k, \tau_1)} e^{\varphi_p(\beta\tau)} \right).$$

We then have

$$(3.6) \quad \mathcal{L}^{k+p}(g)(\omega) - \mathcal{L}^{k+p}(g)(\tau) = \Delta_1 \mathcal{L}^{k+p}(g)(\omega) + \Delta_2 \mathcal{L}^{k+p}(g)(\omega, \tau) - \Delta_1 \mathcal{L}^{k+p}(g)(\tau).$$

We will estimate the absolute value of each of these three summands in terms of  $\omega$  only (i.e. independently of  $\tau$ ) and then we will integrate against the measure  $\mu_\varphi$ . First:

$$\begin{aligned}
(3.7) \quad |\Delta_1 \mathcal{L}^{k+p}(g)(\rho)| &\leq \sum_{|\gamma|=k} \sum_{\alpha \in E_A^p(\gamma_k, \rho_1)} |g(\gamma\alpha\rho) e^{\varphi_k(\gamma\alpha\rho)} - \hat{g}(\gamma) e^{\hat{\varphi}_k(\gamma)}| e^{\varphi_p(\alpha\rho)} \\
&\leq \sum_{|\gamma|=k} \sum_{\alpha \in E_A^p(\gamma_k, \rho_1)} (|g(\gamma\alpha\rho) - \hat{g}(\gamma)| e^{\varphi_{k+p}(\gamma\alpha\rho)} + |e^{\varphi_k(\gamma\alpha\rho)} - e^{\hat{\varphi}_k(\gamma)}| \cdot |\hat{g}(\gamma)| e^{\varphi_p(\alpha\rho)}) \\
&\leq \sum_{|\gamma|=k} \sum_{\alpha \in E_A^p(\gamma_k, \rho_1)} (\text{osc}_k(g|_{[\gamma]}) e^{\varphi_{k+p}(\gamma\alpha\rho)} + M_\varphi e^{\varphi_k(\gamma\alpha\rho)} e^{\varphi_p(\alpha\rho)} |\hat{g}(\gamma)|) \\
&\leq \sum_{|\gamma|=k} \sum_{\alpha \in E_A^p(\gamma_k, \rho_1)} \text{osc}_k(g|_{[\gamma]}) e^{\varphi_{k+p}(\gamma\alpha\rho)} + M_\varphi \sum_{|\gamma|=k} \sum_{\alpha \in E_A^p(\gamma_k, \rho_1)} |g(\gamma\alpha\rho)| e^{\varphi_{k+p}(\gamma\alpha\rho)} \\
&= \mathcal{L}^{k+p}(\text{osc}_k(g))(\rho) + M_\varphi \mathcal{L}^{k+p}(|g|)(\rho),
\end{aligned}$$

with some appropriately large constant  $M_\varphi$ . Plugging into the above inequality,  $\rho = \omega$ , this gives

$$(3.8) \quad |\Delta_1 \mathcal{L}^{k+p}(g)(\omega)| \leq \mathcal{L}^{k+p}(\text{osc}_k(g))(\omega) + M_\varphi \mathcal{L}^{k+p}(|g|)(\omega).$$

Now notice that because of our choice of  $p \geq 1$  there exists a number  $Q \geq 1$  and for every  $e \in E$  there exists an at most  $Q$ -to-1 function  $f_e : E_A^p(e, \tau_1) \rightarrow E_A^p(e, \omega_1)$  (can be chosen to be a bijection if the alphabet  $E$  is infinite). So, plugging in turn  $\rho = \tau$  to (3.7), we get

$$\begin{aligned}
(3.9) \quad |\Delta_1 \mathcal{L}^{k+p}(g)(\tau)| &\leq \sum_{|\gamma|=k} \sum_{\beta \in E_A^p(\gamma_k, \tau_1)} \text{osc}_k(g|_{[\gamma]}) e^{\varphi_{k+p}(\gamma\beta\tau)} + M_\varphi \sum_{|\gamma|=k} \sum_{\beta \in E_A^p(\gamma_k, \tau_1)} |g(\gamma\beta\tau)| e^{\varphi_{k+p}(\gamma\beta\tau)} \\
&\leq M_\varphi \sum_{|\gamma|=k} \sum_{\beta \in E_A^p(\gamma_k, \tau_1)} (\text{osc}_k(g)(\gamma f_e(\beta)\omega) e^{\varphi_{k+p}(\gamma f_e(\beta)\omega)} + M_\varphi |\hat{g}(\gamma)| e^{\varphi_{k+p}(\gamma f_e(\beta)\omega)}) \\
&\leq M_\varphi \sum_{|\gamma|=k} \sum_{\beta \in E_A^p(\gamma_k, \tau_1)} \text{osc}_k(g)(\gamma f_e(\beta)\omega) e^{\varphi_{k+p}(\gamma f_e(\beta)\omega)} + M_\varphi \sum_{|\gamma|=k} \sum_{\beta \in E_A^p(\gamma_k, \tau_1)} |g(\gamma f_e(\beta)\omega)| e^{\varphi_{k+p}(\gamma f_e(\beta)\omega)} \\
&\leq Q M_\varphi \left( \sum_{|\gamma|=k} \sum_{\alpha \in E_A^p(\gamma_k, \omega_1)} \text{osc}_k(g)(\gamma\alpha\omega) e^{\varphi_{k+p}(\gamma\alpha\omega)} + M_\varphi \sum_{|\gamma|=k} \sum_{\alpha \in E_A^p(\gamma_k, \omega_1)} |g(\gamma\alpha\omega)| e^{\varphi_{k+p}(\gamma\alpha\omega)} \right) \\
&= Q M_\varphi (\mathcal{L}^{k+p}(\text{osc}_k(g))(\omega) + M_\varphi \mathcal{L}^{k+p}(|g|)(\omega))
\end{aligned}$$

with some appropriate constant  $Q > 0$ . Turning to  $\Delta_2 \mathcal{L}^{k+p}(g)$ , we get

$$\begin{aligned}
|\Delta_2 \mathcal{L}^{k+p}(g)(\omega, \tau)| &\leq \sum_{|\gamma|=k} |\hat{g}(\gamma)| e^{\hat{\varphi}_k(\gamma)} \left( \sum_{\alpha \in E_A^p(\gamma_k, \omega_1)} e^{\varphi_p(\alpha\omega)} + \sum_{\beta \in E_A^p(\gamma_k, \tau_1)} e^{\varphi_p(\beta\tau)} \right) \\
&\leq \sum_{|\gamma|=k} |\hat{g}(\gamma)| e^{\hat{\varphi}_k(\gamma)} (\mathcal{L}^p \mathbb{1}(\omega) + \mathcal{L}^p \mathbb{1}(\tau)) \\
(3.10) \quad &= 2 \sum_{|\gamma|=k} |\hat{g}(\gamma)| e^{\hat{\varphi}_k(\gamma)} \\
&\leq 2M_\varphi \sum_{|\gamma|=k} |g(\gamma\alpha(\gamma_k, \omega_1)\omega)| e^{\varphi_{k+p}(\gamma\alpha(\gamma_k, \omega_1)\omega)} e^{-\varphi_p(\alpha(\gamma_k, \omega_1)\omega)} \\
&\leq 2M_\varphi e^{-C_p} \sum_{|\gamma|=k} |g(\gamma\alpha(\gamma_k, \omega_1)\omega)| e^{\varphi_{k+p}(\gamma\alpha(\gamma_k, \omega_1)\omega)} \\
&\leq 2M_\varphi e^{-C_p} \mathcal{L}^{k+p}(|g|)(\omega),
\end{aligned}$$

where  $\alpha(\gamma_k, \omega_1)$  is one, arbitrarily chosen, element from  $\Lambda$ , a finite set witnessing finite primitivity of  $A$ , such that  $\gamma\alpha(\gamma_k, \omega_1) \in E_A^*$ , and  $C_p := \min\{\inf\{\varphi_p|_{[\alpha]} : \alpha \in \Lambda\} > 0$ . Inserting now (3.10), (3.9), and (3.8) to (3.6), we get for all  $\omega, \tau \in E_A^\infty$  that

$$|\mathcal{L}^{k+p}(g)(\omega) - \mathcal{L}^{k+p}(g)(\tau)| \leq C(\mathcal{L}^{k+p}(\text{osc}_k(g))(\omega) + \mathcal{L}^{k+p}(|g|)(\omega))$$

with some universal constant  $C > 0$ . Integrating against the measure  $\mu_\varphi$ , this gives

$$\begin{aligned}
\theta^{-0} \|\text{osc}_0(\mathcal{L}^{k+p}(g))\|_1 &\leq C \left( \int_{E_A^\infty} \mathcal{L}^{k+p}(\text{osc}_k(g)) d\mu_\varphi + \int_{E_A^\infty} \mathcal{L}^{k+p}(|g|) d\mu_\varphi \right) \\
(3.11) \quad &= C \left( \int_{E_A^\infty} \text{osc}_k(g) d\mu_\varphi + \int_{E_A^\infty} |g| d\mu_\varphi \right) \\
&\leq C(\theta^k |g|_\theta + \|g\|_1) \\
&\leq C\theta^{-p}(\theta^{k+p} |g|_\theta + \|g\|_1).
\end{aligned}$$

Along with (3.5) this gives that

$$(3.12) \quad |\mathcal{L}^k g|_\theta \leq C(\theta^k |g|_\theta + \|g\|_1)$$

for all  $k \geq p$  with some suitable constant  $C > 0$ . Also, for every  $0 \leq k \leq p$  we have

$$\begin{aligned}
|\mathcal{L}^k g|_\theta &\leq \|\mathcal{L}^k g\|_\theta \leq \max\{\|\mathcal{L}\|_\theta^j : 0 \leq j \leq p\} \|g\|_\theta \\
&\leq \theta^{-p} \max\{\|\mathcal{L}\|_\theta^j : 0 \leq j \leq p\} \|g\|_\theta (\theta^k |g|_\theta + \|g\|_1) \\
&\leq \theta^{-p} \max\{\|\mathcal{L}\|_\theta^j : 0 \leq j \leq p\} \|g\|_\theta (\theta^k |g|_\theta + \|g\|_1).
\end{aligned}$$

Along with (3.12) this finishes the proof.  $\square$

#### 4. SINGULAR PERTURBATIONS OF (ORIGINAL) PERRON–FROBENIUS OPERATORS I: FUNDAMENTAL INEQUALITIES

This is the first section in which we deal with singular perturbations of the operator  $\mathcal{L}_\varphi$ . We work in the quite general setting described below. We keep the same non-standard Banach space  $\mathcal{B}_\theta$  but, motivated by [8], we introduce an even more exotic norm  $\|\cdot\|_*$ , which depends even more on dynamics than  $\|\cdot\|_\theta$ .

Passing to details, in this section we assume that  $(U_n)_{n=0}^\infty$ , a nested sequence of open subsets of  $E_A^\infty$  is given, with the following properties:

- (U0)  $U_0 = E_A^\infty$ ,
- (U1) For every  $n \geq 0$  the open set  $U_n$  is a (disjoint) union of cylinders all of which are of length  $n$ ,
- (U2) There exists  $\rho \in (0, 1)$  such that

$$\mu_\varphi(U_n) \leq \rho^n$$

for all  $n \geq 0$ .

Let  $|\cdot|_*$ ,  $\|\cdot\|_* : \mathcal{B}_\theta \rightarrow [0, +\infty]$  be the functions defined by respective formulas

$$|g|_* := \sup_{j \geq 0} \sup_{m \geq 0} \left\{ \theta^{-m} \int_{\sigma^{-j}(U_m)} |g| d\mu_\varphi \right\}$$

and

$$\|g\|_* := \|g\|_1 + |g|_*.$$

Without loss of generality assume from now on that  $\theta \in (\rho, 1)$ . We shall prove the following.

**Lemma 4.1.** *For all  $g \in \mathcal{B}_\theta$ , we have that*

$$\|g\|_* \leq 2\|g\|_\infty \leq 2\|g\|_\theta.$$

*Proof.* By virtue of (U2), we get

$$|g|_* \leq \sup_{m \geq 0} \left\{ \theta^{-m} \mu_\varphi(U_m) \|g\|_\infty \right\} \leq \sup_{m \geq 0} \left\{ \theta^{-m} \rho^m \|g\|_\infty \right\} = \sup_{m \geq 0} \left\{ (\rho/\theta)^m \|g\|_\infty \right\} = \|g\|_\infty.$$

Hence,

$$\|g\|_* = \|g\|_1 + |g|_* \leq \|g\|_\infty + \|g\|_\infty = 2\|g\|_\infty.$$

Combining this with Lemma 3.1 completes the proof.  $\square$

In particular, this lemma assures us that  $|\cdot|_*$  and  $\|\cdot\|_*$ , respectively, are a semi-norm and a norm on  $\mathcal{B}_\theta$ . It is straightforward to check that  $\mathcal{B}_\theta$  endowed with the norm  $\|\cdot\|_*$  becomes a Banach space. For all integers  $k \geq 1$  and  $n \geq 0$  let

$$(4.1) \quad \mathbb{1}_n^k := \prod_{j=0}^{k-1} \mathbb{1}_{\sigma^{-j}(U_n^c)} = \prod_{j=0}^{k-1} \mathbb{1}_{U_n^c} \circ \sigma^j.$$

We also abbreviate

$$\mathbb{1}_n := \mathbb{1}_n^1$$

and set

$$\mathbb{1}_n^c := \mathbb{1}_{U_n} = \mathbb{1} - \mathbb{1}_n.$$

Let  $\mathcal{L}_n : \mathcal{B}_\theta \rightarrow \mathcal{B}_\theta$  be defined by the formula

$$\mathcal{L}_n(g) := \mathcal{L}(\mathbb{1}_n^1 g).$$

These, for  $n \geq 0$ , are our perturbations of the operator  $\mathcal{L}$ . The difference  $\mathcal{L} - \mathcal{L}_n$  in the supremum, or even  $\|\cdot\|_\theta$ , norm can be quite large even for arbitrarily large  $n$ , however, as Lemma 5.1 shows, the incorporation of the  $\|\cdot\|_*$  norm makes this difference kind of small. The main result of this section is Proposition 5.2, complemented by Proposition 5.3, which describes in detail how well the spectral properties of the operator  $\mathcal{L}$  are preserved under perturbations  $\mathcal{L}_n$ . Note that for every  $k \geq 1$ , we then have

$$\mathcal{L}_n^k(g) := \mathcal{L}^k(\mathbb{1}_n^k g).$$

The results we now obtain, leading ultimately to Proposition 5.2 and Proposition 5.3, stem from Lemma 3.9 and Lemma 3.10 in [8]. We develop these and extend them to the case of infinite alphabets. Since the sets  $U_n$  may, and in applications, will, consist of infinitely many cylinders (of the same length), we are cannot take advantage of good mixing properties of the symbol dynamical system  $(\sigma : E_A^\infty \rightarrow E_A^\infty, \mu_\varphi)$ . We use instead the Hölder inequality, which also, as a by-product, simplifies some of the reasoning of [8]. In what follows, the last fragment, directly preceding Proposition 5.2, and leading to verifying the requirements from Remark 3 in [11], is particularly delicate and entirely different to [8].

**Lemma 4.2.** *For all integers  $k \geq 1$  and  $n \geq 0$ , we have*

$$\|\mathcal{L}_n^k\|_* \leq 1.$$

*Proof.* Let  $g \in L^1(\mu_\varphi)$ . Then,

$$(4.2) \quad \|\mathcal{L}_n^k(g)\|_1 = \int |\mathcal{L}^k(\mathbb{1}_n^k g)| d\mu_\varphi \leq \int \mathcal{L}^k(|\mathbb{1}_n^k g|) d\mu_\varphi = \int |\mathbb{1}_n^k g| d\mu_\varphi \leq \|g\|_1.$$

Also, for all integers  $j, m \geq 0$ , we have

$$\begin{aligned} \theta^{-m} \int_{\sigma^{-j}(U_m)} |\mathcal{L}^k(\mathbb{1}_n^k g)| d\mu_\varphi &\leq \theta^{-m} \int_{\sigma^{-j}(U_m)} \mathcal{L}^k(|\mathbb{1}_n^k g|) d\mu_\varphi = \theta^{-m} \int_{\sigma^{-(j+1)}(U_m)} |\mathbb{1}_n^k g| d\mu_\varphi \\ &\leq \theta^{-m} \int_{\sigma^{-(j+1)}(U_m)} |g| d\mu_\varphi \\ &\leq \|g\|_*. \end{aligned}$$

Taking the supremum over  $j$  and  $m$  yields

$$|\mathcal{L}_n^k(g)|_* \leq \|g\|_*.$$

Combining this and (4.2) completes the proof.  $\square$

**Lemma 4.3.** *For all integers  $j, n \geq 0$  and for  $g \in \mathcal{B}_\theta$ , we have that*

$$|g \mathbb{1}_{\sigma^{-j}(U_n^c)}|_\theta \leq \|g\|_\theta + \theta^{-j} \|g\|_*$$

*Proof.* Fix an integer  $m \geq 1$ . We consider two cases. Namely:  $j + n \leq m$  and  $m < j + n$ . Suppose first that  $j + n \leq m$ . Then,  $\text{osc}_m(g \mathbb{1}_{\sigma^{-j}(U_n^c)})(\omega) \leq \text{osc}_m(g)(\omega)$  for all  $\omega \in E_A^\infty$ . Thus

$$(4.3) \quad \theta^{-m} \int \text{osc}_m(g \mathbb{1}_{\sigma^{-j}(U_n^c)}) d\mu_\varphi \leq \theta^{-m} \int \text{osc}_m(g) d\mu_\varphi \leq |g|_\theta.$$

On the other hand, if  $m < j + n$ , then it is easy to see that if  $[\omega|_m] \subseteq \sigma^{-j}(U_n^c)$ , then

$$(4.4) \quad \text{osc}_m(g \mathbb{1}_{\sigma^{-j}(U_n^c)})(\omega) = \text{osc}_m(g)(\omega).$$

On the other hand, if  $[\omega|_m] \cap \sigma^{-j}(U_n) \neq \emptyset$ , then

$$\text{osc}_m(g \mathbb{1}_{\sigma^{-j}(U_n^c)})(\omega) = \max\{\text{osc}_m(g)(\omega), \|g \mathbb{1}_{[\omega|_m]}\|_\infty\}.$$

In this latter case

$$\text{osc}_m(g \mathbb{1}_{\sigma^{-j}(U_n^c)}) \leq \max\{\text{osc}_m(g)(\omega), \|g \mathbb{1}_{[\omega|_m]}\|_\infty\} \leq \text{osc}_m(g)(\omega) + \frac{1}{\mu_\varphi([\omega|_m])} \int_{[\omega|_m]} |g| d\mu_\varphi.$$

Together with (4.4) this implies that

$$(4.5) \quad \theta^{-m} \int \text{osc}_m(g \mathbb{1}_{\sigma^{-j}(U_n^c)}) d\mu_\varphi \leq |g|_\theta + \theta^{-m} \int_{\{\omega \in E_A^\infty : [\omega|_m] \cap \sigma^{-j}(U_n) \neq \emptyset\}} |g| d\mu_\varphi.$$

We now consider two further sub-cases. If  $m \leq j$ , then we see that

$$(4.6) \quad \theta^{-m} \int_{\{\omega \in E_A^\infty : [\omega|_m] \cap \sigma^{-j}(U_n) \neq \emptyset\}} |g| d\mu_\varphi \leq \theta^{-j} \int_{\{\omega \in E_A^\infty : [\omega|_m] \cap \sigma^{-j}(U_n) \neq \emptyset\}} |g| d\mu_\varphi \leq \theta^{-j} \|g\|_1.$$

If  $j < m < j + n$ , the descending property of the sequence  $(U_n)_{n=0}^\infty$  yields

$$\{\omega \in E_A^\infty : [\omega|_m] \cap \sigma^{-j}(U_n) \neq \emptyset\} \subseteq \sigma^{-j}(U_{m-j}).$$

In this case

$$(4.7) \quad \theta^{-m} \int_{\{\omega \in E_A^\infty : [\omega|_m] \cap \sigma^{-j}(U_n) \neq \emptyset\}} |g| d\mu_\varphi \leq \theta^{-j} \theta^{-(m-j)} \int_{\sigma^{-j}(U_{m-j})} |g| d\mu_\varphi \leq \theta^{-j} |g|_*.$$

Combining (4.3), (4.5), and (4.7) yields the desired inequality, and completes the proof.  $\square$

As a fairly straightforward inductive argument using Lemma 4.3, we shall prove the following.

**Lemma 4.4.** *For all integers  $k \geq 1$  and  $n \geq 0$ , and all functions  $g \in \mathcal{B}_\theta$ , we have that*

$$(4.8) \quad |\mathbb{1}_n^k g|_\theta \leq |g|_\theta + \theta(1 - \theta)^{-1} \theta^{-k} \|g\|_*.$$

*Proof.* Keeping  $n \geq 0$  fixed, we will proceed by induction with respect to the integer  $k \geq 1$ . The case of  $k = 1$  follows directly from Lemma 4.3. Assuming for the inductive step that



(4.8) for some integer  $k \geq 1$  and applying again Lemma 4.3, we get

$$\begin{aligned}
|\mathbb{1}_n^{k+1}g|_\theta &= |\mathbb{1}_{\sigma^{-k}(U_n^c)}(\mathbb{1}_n^k g)|_\theta \leq |\mathbb{1}_n^k g|_\theta + \theta^{-k} \|\mathbb{1}_n^k g\|_* \\
&\leq |\mathbb{1}_n^k g|_\theta + \theta^{-k} \|g\|_* \\
&\leq |g|_\theta + \theta(1-\theta)^{-1} \theta^{-k} \|g\|_* + \theta^{-k} \|g\|_* \\
&= |g|_\theta + \theta(1-\theta)^{-1} \theta^{-(k+1)} \|g\|_*.
\end{aligned}$$

The proof is complete.  $\square$

As a fairly immediate consequence of Lemma 4.4 and Lemma 3.2, we get the following.

**Corollary 4.5.** *There exists a constant  $c > 0$  such that*

$$\|\mathcal{L}_n^k g\|_\theta \leq c(\theta^k \|g\|_\theta + \|g\|_*)$$

for all  $g \in \mathcal{B}_\theta$  and all integers  $k, n \geq 0$ .

*Proof.* Substituting  $\mathbb{1}_n^k g$  for  $g$  into the statement of Lemma 3.2 and then applying Lemma 4.3, we get

$$\begin{aligned}
|\mathcal{L}_n^k g|_\theta &= |\mathcal{L}^k(\mathbb{1}_n^k g)|_\theta \leq C(\theta^k |\mathbb{1}_n^k g|_\theta + \|g\|_1) \\
&\leq C(\theta^k (|g|_\theta + \theta(1-\theta)^{-1} \theta^{-k} \|g\|_*) + \|g\|_1) \\
&\leq C(\theta^k (|g|_\theta + \theta(1-\theta)^{-1} \|g\|_* + \|g\|_1)).
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\mathcal{L}_n^k g\|_\theta &= |\mathcal{L}_n^k g|_\theta + \|\mathcal{L}_n^k g\|_1 \leq |\mathcal{L}_n^k g|_\theta + \|g\|_1 \\
&\leq (C+1)(\theta^k |g|_\theta + \theta(1-\theta)^{-1} \|g\|_* + \|g\|_1) \\
&\leq \tilde{C}(\theta^k \|g\|_\theta + \|g\|_*),
\end{aligned}$$

for some sufficiently large  $\tilde{C} > 0$  depending only on  $C$  and  $\theta$ . The proof is complete.  $\square$

## 5. SINGULAR PERTURBATIONS OF (ORIGINAL) PERRON-FROBENIUS OPERATORS II: STABILITY OF THE SPECTRUM

For a linear operator  $Q : \mathcal{B}_\theta \rightarrow \mathcal{B}_\theta$  define

$$|||Q||| := \sup\{\|Qg\|_* : \|g\|_\theta \leq 1\}.$$

From now on fix  $p, q > 1$  such that  $\frac{1}{p} + \frac{1}{q} = 1$  and, by taking  $0 < \rho < 1$  coming from (U2), sufficiently close to 1, assume without loss of generality that

$$\theta \in (\rho^{1/p}, 1).$$

We shall prove the following.

**Lemma 5.1.** *For every  $n \geq 0$  we have*

$$|||\mathcal{L} - \mathcal{L}_n||| \leq 2(\rho^{1/q})^n.$$

*Proof.* Fix an arbitrary  $g \in \mathcal{B}_\theta$  with  $\|g\|_\theta \leq 1$ . Using Lemma 3.1 we then get

$$\begin{aligned}
 \|(\mathcal{L} - \mathcal{L}_n)g\|_1 &= \|\mathcal{L}(\mathbb{1}_n^1 g)\|_1 = \|\mathbb{1}_n^1 g\|_1 \leq \mu_\varphi(U_n) \|g\|_\infty \\
 &\leq \mu_\varphi(U_n) \|g\|_\theta \leq \mu_\varphi(U_n) \leq \rho^n \\
 &\leq (\rho^{1/q})^n
 \end{aligned}
 \tag{5.1}$$

Now fix also two integers  $m, j \geq 0$ . Using the Hölder Inequality, we get

$$\begin{aligned}
 \theta^{-m} \int_{\sigma^{-j}(U_m)} |(\mathcal{L} - \mathcal{L}_n)g| d\mu_\varphi &\leq \theta^{-m} \mu_\varphi(\sigma^{-(j+1)}(U_m) \cap U_n) \|g\|_\theta = \\
 &= \theta^{-m} \|g\|_\theta \int \mathbb{1}_{\sigma^{-(j+1)}(U_m)} \mathbb{1}_{U_n} d\mu_\varphi \\
 &\leq \|g\|_\theta \theta^{-m} \left( \int \mathbb{1}_{\sigma^{-(j+1)}(U_m)} d\mu_\varphi \right)^{1/p} \left( \int \mathbb{1}_{U_n} d\mu_\varphi \right)^{1/q} \\
 &= \|g\|_\theta \theta^{-m} \mu_\varphi(U_m)^{1/p} \mu_\varphi(U_n)^{1/q} \\
 &\leq \|g\|_\theta (\rho^{1/p}/\theta)^m \rho^{n/q} \leq (\rho^{1/q})^n \|g\|_\theta \leq (\rho^{1/q})^n,
 \end{aligned}
 \tag{5.2}$$

where the second to the last inequality follows from the fact that  $\theta \in (\rho^{1/p}, 1)$ . Along with (5.1) this implies that  $\|\mathcal{L} - \mathcal{L}_n\|_* \leq 2(\rho^{1/q})^n$ . So, taking the supremum over all  $g \in \mathcal{B}_\theta$  with  $\|g\|_\theta \leq 1$ , we get that  $\|\mathcal{L} - \mathcal{L}_n\| \leq 2(\rho^{1/q})^n$ . The proof is complete.  $\square$

With Lemma 4.2, Corollary 4.5, and Lemma 4.3, we have checked that the respective conditions (2), (3), and (5), from [11] are satisfied. We shall now check that condition (4) from there also holds. We will do this by showing that the requirements from Remark 3 in [11] hold.

For every integer  $k \geq 1$  let  $\mathcal{A}^k$  be the partition of  $E_A^\infty$  into cylinders of length  $k$ . Let  $\pi_k^* : L^1(\mu_\varphi) \rightarrow L^1(\mu_\varphi)$  be the operator of expected value with respect to the probability measure  $\mu_\varphi$  and the  $\sigma$ -algebra  $\sigma(\mathcal{A}^k)$  generated by the elements of  $\mathcal{A}^k$ ; i. e.

$$\pi_k^*(g) = E_{\mu_\varphi}(g | \sigma(\mathcal{A}^k)).$$

If  $g \in \mathcal{B}_\theta$  then  $|\pi_k^*(g) - g| \leq \text{osc}_k(g)$ , and therefore

$$\|\pi_k^*(g) - g\|_1 = \int_{E_A^\infty} |\pi_k^*(g) - g| d\mu_\varphi \leq \int_{E_A^\infty} \text{osc}_k(g) d\mu_\varphi \leq \theta^k \|g\|_\theta.
 \tag{5.3}$$

Let now  $\mathcal{A}_0^k$  be a finite subset of  $\mathcal{A}^k$  such that

$$\mu_\varphi(A_c^k) \leq \theta^k,
 \tag{5.4}$$

where

$$A_c^k := \bigcup_{A \in \mathcal{A}^k \setminus \mathcal{A}_0^k} A.$$

Let also

$$A_0^k := \bigcup_{A \in \mathcal{A}_0^k} A.$$

Let  $\hat{\mathcal{A}}^k$  be the partition of  $E_A^\infty$  consisting of  $A_c^k$  and all elements of  $\mathcal{A}_0^k$ . Similarly as above, let  $\pi_k : L^1(\mu_\varphi) \rightarrow L^1(\mu_\varphi)$  be defined by the formula

$$\pi_k(g) = E_{\mu_\varphi}(g|\sigma(\hat{\mathcal{A}}^k)).$$

We then have that

$$(5.5) \quad \|\pi_k\|_1 \leq 1,$$

and for every  $g \in \mathcal{B}_\theta$ , because of (5.3) and Lemma 3.1, and (5.4):

$$\begin{aligned} \|\pi_k(g) - g\|_1 &= \int_{E_A^\infty} |\pi_k(g) - g| d\mu_\varphi = \int_{A_0^k} |\pi_k(g) - g| d\mu_\varphi + \int_{A_c^k} |\pi_k(g) - g| d\mu_\varphi \\ &= \int_{A_0^k} |\pi_k^*(g) - g| d\mu_\varphi + \int_{A_c^k} |\pi_k(g) - g| d\mu_\varphi \\ (5.6) \quad &\leq \int_{E_A^\infty} |\pi_k^*(g) - g| d\mu_\varphi + 2\|g\|_\infty \mu_\varphi(A_c^k) \\ &\leq \theta^k |g|_\theta + 2\|g\|_\infty \theta^k \\ &\leq 3\theta^k \|g\|_\theta. \end{aligned}$$

Now, for all  $m$  and  $k$  we have that

$$\text{osc}_m(\pi_k(g)) = \begin{cases} 0 & \text{if } m \geq k \\ \leq \text{osc}_0(g) \leq 2\|g\|_\infty \leq 2\|g\|_\theta & \text{if } m < k. \end{cases}$$

Moreover, if  $\omega \in A_0^k$  and  $m < k$ , then

$$\text{osc}_m(\pi_k(g))(\omega) = \text{osc}_m(\pi_k^*(g))(\omega) \leq \text{osc}_m(g)(\omega).$$

Thus,

$$\begin{aligned} \theta^{-m} \|\text{osc}_m(\pi_k(g))\|_1 &= \theta^{-m} \int_{E_A^\infty} \text{osc}_m(\pi_k(g)) d\mu_\varphi \\ &= \theta^{-m} \int_{A_0^k} \text{osc}_m(\pi_k(g)) d\mu_\varphi + \theta^{-m} \int_{A_c^k} \text{osc}_m(\pi_k(g)) d\mu_\varphi \\ &\leq \theta^{-m} \int_{A_0^k} \text{osc}_m(g) d\mu_\varphi + 2\theta^{-k} \|g\|_\theta \mu_\varphi(A_c^k) \\ &\leq |g|_\theta + 2\|g\|_\theta \\ &\leq 3\|g\|_\theta. \end{aligned}$$

Therefore  $|\pi_k(g)|_\theta \leq 3\|g\|_\theta$ . Together with (5.5), this gives  $\|\pi_k\|_\theta \leq 4$ . In other words:

$$(5.7) \quad \sup_{k \geq 1} \{\|\pi_k\|_\theta\} \leq 4 < +\infty.$$

Now assume that  $\|g\|_\theta \leq 1$ . Recall that we have fixed  $p, q > 1$  such that  $(1/p) + (1/q) = 1$ . Using Hölder's Inequality and (5.6) we then get for all integers  $k \geq 1$ ,  $j \geq 0$ , and  $n \geq 0$ , that

$$\begin{aligned} \int_{\sigma^{-j}(U_n)} |\pi_k(g) - g| d\mu_\varphi &= \int_{E_A^\infty} \mathbb{1}_{\sigma^{-j}(U_n)} |\pi_k(g) - g| d\mu_\varphi \\ &\leq \left( \int_{E_A^\infty} \mathbb{1}_{\sigma^{-j}(U_n)} d\mu_\varphi \right)^{1/p} \left( \int_{E_A^\infty} |\pi_k(g) - g|^q d\mu_\varphi \right)^{1/q} \\ &\leq \mu_\varphi(U_n)^{1/p} 2^{\frac{q-1}{q}} \left( \int_{E_A^\infty} |\pi_k(g) - g| d\mu_\varphi \right)^{1/q} \\ &\leq \mu_\varphi(U_n)^{1/p} (3\theta^k \|g\|_\theta)^{1/q} \\ &\leq 3\rho^{n/p} \theta^{k/q}. \end{aligned}$$

Recall that  $\theta \in (0, 1)$  was fixed so large that  $\theta > \rho^{1/p}$ . In other words  $\rho^{1/p}/\theta < 1$ , and we get

$$\theta^{-n} \int_{\sigma^{-j}(U_n)} |\pi_k(g) - g| d\mu_\varphi \leq 3(\rho^{1/p}/\theta)^n \theta^{k/q} \leq 3\theta^{k/q}.$$

In other words  $\|\pi_k(g) - g\|_* \leq 3\theta^{k/q}$ . Together with (5.6) this gives

$$(5.8) \quad |\pi_k(g) - g|_* \leq 2\theta^k + 3\theta^{k/q} \leq 5(\theta^{1/q})^k.$$

Since all the operators  $\pi_k : \mathcal{B}_\theta \rightarrow \mathcal{B}_\theta$  have finite-dimensional ranges, all the operators  $\mathcal{L}_n \circ \pi_k : \mathcal{B}_\theta \rightarrow \mathcal{B}_\theta$  are compact. As  $\theta \leq \theta^{1/q}$ , in conjunction with (5.8) and (5.7), this shows that all the requirements of Remark 3 in [11] are satisfied and (4) (as well as (3)) hold with  $\alpha = \theta^{1/q}$  and  $M = 1$ . All the hypotheses of Theorem 1 in [11] have been thus verified. Note also that the number 1 is a simple eigenvalue of the operator  $\mathcal{L} : \mathcal{B}_\theta \rightarrow \mathcal{B}_\theta$  as there exists exactly one Borel probability  $\sigma$ -invariant measure absolutely continuous with respect to the Gibbs measure  $\mu_\varphi$ . Applying Theorem 1 in [11] and the appropriate corollaries therein, we get the following fundamental perturbative result which extends Propositions 3.17, 3.19, and 3.7 from [8] to the case of infinite alphabet.

**Proposition 5.2.** *For all  $n \geq 0$  sufficiently large there exist two bounded linear operators  $Q_n, \Delta_n : \mathcal{B}_\theta \rightarrow \mathcal{B}_\theta$  and complex numbers  $\lambda_n \neq 0$  with the following properties:*

- (a)  $\lambda_n$  is a simple eigenvalue of the operator  $\mathcal{L}_n : \mathcal{B}_\theta \rightarrow \mathcal{B}_\theta$ .
- (b)  $Q_n : \mathcal{B}_\theta \rightarrow \mathcal{B}_\theta$  is a projector ( $Q_n^2 = Q_n$ ) onto the 1-dimensional eigenspace of  $\lambda_n$ .
- (c)  $\mathcal{L}_n = \lambda_n Q_n + \Delta_n$ .

(d)  $Q_n \circ \Delta_n = \Delta_n \circ Q_n = 0$ .

(e) *There exist  $\kappa \in (0, 1)$  and  $C > 0$  such that*

$$\|\Delta_n^k\|_\theta \leq C\kappa^k$$

*for all  $n \geq 0$  sufficiently large and all  $k \geq 0$ . In particular,*

$$\|\Delta_n^k g\|_\infty \leq \|\Delta_n^k g\|_\theta \leq C\kappa^k \|g\|_\theta$$

*for all  $g \in \mathcal{B}_\theta$ .*

(f)  $\lim_{n \rightarrow \infty} \lambda_n = 1$ .

(g) *Enlarging the above constant  $C > 0$  if necessary, we have*

$$\|Q_n\|_\theta \leq C.$$

*In particular,*

$$\|Q_n g\|_\infty \leq \|Q_n g\|_\theta \leq C \|g\|_\theta$$

*for all  $g \in \mathcal{B}_\theta$ .*

(h)  $\lim_{n \rightarrow \infty} \|Q_n - Q\| = 0$ .

The proof of the next proposition is fairly standard. We provide it here for the sake of completeness.

**Proposition 5.3.** *All eigenvalues  $\lambda_n$  produced in Proposition 5.2 are real and positive, and all operators  $Q_n : \mathcal{B}_\theta \rightarrow \mathcal{B}_\theta$  preserve  $\mathcal{B}_\theta(\mathbb{R})$  and  $\mathcal{B}_\theta^+(\mathbb{R})$ , the subsets of  $\mathcal{B}_\theta$  consisting, respectively, of real-valued functions and positive real-valued functions.*

*Proof.* Let  $\rho_n \in \mathcal{B}_\theta$  be an eigenfunction of the eigenvalue  $\lambda_n$ . Write  $\lambda_n = |\lambda_n|e^{i\gamma_n}$ , with  $\gamma_n \in [0, 2\pi)$ . It follows from (b), (c), and (d) of Proposition 5.2 that

$$(5.9) \quad |\lambda_n|^k e^{-ik\gamma_n} \mathcal{L}_n^k \mathbb{1} = Q_n \mathbb{1} + \lambda_n^{-k} \Delta_n^k \mathbb{1}.$$

By (1) of Corollary 1 in [11] we have that  $Q_n \mathbb{1} \neq 0$  for all  $n \geq 0$  large enough (so after disregarding finitely many terms, we can assume this for all  $n \geq 0$ ) and  $|\lambda_n| > (1 + \kappa)/2$ . Since also  $\mathcal{L}_n^k \mathbb{1}$  is a real-valued function, it therefore follows from (5.9) and (e) that the arguments of  $Q_n \mathbb{1}(\omega)$  are the same (mod  $2\pi$ ) whenever  $Q_n \mathbb{1} \neq 0$ . This in turn implies that the set of accumulation points of the sequence  $(k\gamma_n)_{k=0}^\infty$  is a singleton (mod  $2\pi$ ). This yields  $\gamma_n = 0$  (mod  $2\pi$ ). Thus  $\lambda_n \in \mathbb{R}$ , and, as  $\lambda_n$  is close to 1 (by Proposition 5.2), it is positive. Knowing this and assuming  $g \geq 0$ , the equality

$$Q_n g = \lambda_n^{-k} \mathcal{L}_n^k g - \lambda_n^{-k} \Delta_n^k(g),$$

along with (e) of Proposition 5.2, non-negativity of  $\mathcal{L}_n^k g$ , and inequality  $|\lambda_n| > (1 + \kappa)/2$ , yield  $Q_n g \geq 0$ . Finally, for  $g \in \mathcal{B}_\theta(\mathbb{R})$ , write canonically  $g = g_+ - g_-$  with  $g_+, g_- \in \mathcal{B}_\theta^+(\mathbb{R})$  and apply the invariance of  $\mathcal{B}_\theta^+(\mathbb{R})$  under the action of  $\mathcal{L}_n$ . The proof is complete.  $\square$

**Remark 5.4.** We would like to note that unlike [8], we did not use the dynamics (i.e., the interpretation of  $\log \lambda_n$  as some topological pressure) to demonstrate item (f) of Proposition 5.2 and to prove Proposition 5.3. We instead used the full power of the perturbation results from [11]. The dynamical interpretation will eventually emerge, and will be important for us, but not until Section 11. Therein Lemma 11.20 will provide, at least in part, a dynamical interpretation.

## 6. AN ASYMPTOTIC FORMULA FOR $\lambda_n$ S, THE LEADING EIGENVALUES OF PERTURBED OPERATORS

In this section we keep the setting of the previous sections. Our goal here is to establish the asymptotic behavior of eigenvalues  $\lambda_n$  as  $n$  diverges to  $+\infty$ . Let

$$U_\infty := \bigcap_{n=0}^{\infty} \overline{U}_n.$$

In addition to (U0), (U1), and (U2), we now also assume that:

(U3)  $U_\infty$  is a finite set.

(U4) Either  
(U4A)

$$U_\infty \cap \bigcup_{n=1}^{\infty} \sigma^n(U_\infty) = \emptyset$$

or

(U4B)  $U_\infty = \{\xi\}$ , where  $\xi$  is a periodic point of  $\sigma$  of prime period equal to some integer  $p \geq 1$ , the pre-concatenation by the first  $p$  terms of  $\xi$  with elements of  $U_n$  satisfy

$$(6.1) \quad [\xi|_p]U_n \subseteq U_n$$

for all  $n \geq 0$ , and

$$(6.2) \quad \lim_{n \rightarrow \infty} \sup\{|\varphi(\omega) - \varphi(\xi)| : \omega \in U_n\} = 0.$$

(U5) There are no integer  $l \geq 1$ , no sequence  $(\omega^{(n)})_{n=0}^{\infty}$  of points in  $E_A^\infty$ , and no increasing sequence  $(s_n)_{n=0}^{\infty}$  of positive integers with the following properties:

(U5A)

$$\omega^{(n)}, \sigma^l(\omega^{(n)}) \in U_{s_n}$$

for all  $n \geq 0$ ,

(U5B)

$$\varliminf_{n \rightarrow \infty} d_\theta(\omega^{(n)}, U_\infty) \begin{cases} > 0 & \text{if (U4A) holds,} \\ > \theta^l & \text{if (U4B) holds,} \end{cases}$$

(U5C)

$$\overline{\lim}_{n \rightarrow \infty} \sum_{i=1}^l \omega_i^{(n)} < +\infty,$$

for fixed  $l$ , where we identify  $E$  with the natural numbers to give  $\omega_i^{(n)}$  their numerical values.

Having proved all the perturbation results of the previous section, we shall now prove the following analogue of Proposition 4.1 in [8], which is our main result concerning the asymptotic behavior of eigenvalues  $\lambda_n$  as  $n \rightarrow +\infty$ .

**Proposition 6.1.** *With the setting of Sections 3 and 4, assume that (U0)–(U5) hold. Then*

$$\lim_{n \rightarrow \infty} \frac{\lambda - \lambda_n}{\mu_\varphi(U_n)} = \begin{cases} \lambda & \text{if (U4A) holds,} \\ \lambda(1 - \lambda^{-p} e^{\varphi_p(\xi)}) & \text{if (U4B) holds,} \end{cases}$$

where  $\lambda$  and  $\lambda_n$  are respective eigenvalues of original (i. e., we recall, not normalized) operators  $\mathcal{L}$  and  $\mathcal{L}_n$ .

For every integer  $n \geq 0$  let  $\nu_n$  be  $\mu_\varphi$ -conditional measure on  $U_n$ , i. e.:

$$\nu_n := \frac{\mu_\varphi|_{U_n}}{\mu_\varphi(U_n)}.$$

We denote

$$\mathbb{1}_n^c := \mathbb{1}_{U_n} = \mathbb{1} - \mathbb{1}_n^1.$$

We start with the following.

**Lemma 6.2.** *If (U0)–(U4A) and (U5) hold, then*

$$\lim_{n \rightarrow \infty} \frac{\int Q_n(\mathcal{L} \mathbb{1}_n^c) d\nu_n}{\lambda - \lambda_n} = \lim_{n \rightarrow \infty} \int_{E_A^\infty} Q_n \mathbb{1} d\nu_n = 1.$$

*Proof.* Assume without loss of generality that  $\mathcal{L}$  is normalized so that  $\lambda = 1$  and  $\mathcal{L} \mathbb{1} = \mathbb{1}$ . With an aim to prove the first equality, we note that

$$\begin{aligned} \int Q_n(\mathcal{L} \mathbb{1}_n^c) d\nu_n &= \int Q_n(\mathcal{L} \mathbb{1} - \mathcal{L} \mathbb{1}_n^1) d\nu_n = \int Q_n(\mathbb{1} - \mathcal{L}_n \mathbb{1}) d\nu_n \\ &= \int Q_n \mathbb{1} d\nu_n - \int Q_n \mathcal{L}_n \mathbb{1} d\nu_n = \int Q_n \mathbb{1} d\nu_n - \int \mathcal{L}_n Q_n \mathbb{1} d\nu_n \\ &= \int Q_n \mathbb{1} d\nu_n - \lambda_n \int Q_n \mathbb{1} d\nu_n \\ &= (1 - \lambda_n) \int Q_n \mathbb{1} d\nu_n, \end{aligned}$$

using Proposition 5.2. and the first equality is established. Now, fix an arbitrary integer  $k \geq 1$ . For every  $\omega \in U_n$  let

$$(6.3) \quad \sigma_0^{-k}(\omega) := \{\tau \in \sigma^{-k}(\omega) : \exists_{0 \leq j \leq k-1} \sigma^j(\tau) \in U_n\}$$

and

$$(6.4) \quad \sigma_c^{-k}(\omega) := \sigma^{-k}(\omega) \setminus \sigma^{-k}(\omega).$$

If  $\tau \in \sigma_0^{-k}(\omega)$ , then  $\sigma^j(\tau) \in U_n$  for some  $0 \leq j \leq k-1$ . Denote  $\sigma^j(\tau)$  by  $\gamma$ . Then

$$\gamma \in U_n \text{ and } \sigma^{k-j}(\gamma) \in U_n; \quad 1 \leq k-j \leq k.$$

Fix an arbitrary  $M > 0$ . We claim that for all  $n \geq 0$  sufficiently large, say  $n \geq N := N_k(M)$ , we have that

$$(6.5) \quad \sum_{i=1}^{k-j} \gamma_i \geq Mk$$

for any  $\gamma = \sigma^j(\tau)$  for any  $\tau \in \sigma_0^{-k}(\omega)$ . Indeed, seeking a contradiction we assume that there exist an increasing sequence  $(s_n)_0^\infty$  of positive integers, a sequence  $(\gamma^{(n)})_0^\infty \subseteq E_A^\infty$ , and an integer  $l \in [1, k]$  such that

$$(6.6) \quad \gamma^{(n)}, \sigma^l(\gamma^{(n)}) \in U_{s_n},$$

and

$$\sum_{i=1}^l \gamma_i^{(n)} < Mk$$

for all  $n \geq 0$ . It then follows from conditions (U4A) and (U5) that the contrapositive of (U5B) holds, i.e.:

$$\varliminf_{n \rightarrow \infty} d_\theta(\gamma^{(n)}, U_\infty) = 0.$$

Hence, from continuity of the shift map  $\sigma : E_A^\infty \rightarrow E_A^\infty$  and from the finiteness of the set  $U_\infty$  (by (U3)),

$$\varliminf_{n \rightarrow \infty} d_\theta(\sigma^l(\gamma^{(n)}), \sigma^l(U_\infty)) = 0.$$

So, passing to a subsequence, and invoking finiteness of the set  $\sigma^l(U_\infty)$ , we may assume without loss of generality that the sequence  $(\sigma^l(\gamma^{(n)}))_0^\infty$  has a limit, call it  $\beta$ , and then  $\beta \in \sigma^l(U_\infty)$ . But, since the sequence  $(\overline{U}_n)_0^\infty$  is descending, it follows from (6.6) that  $\beta \in \overline{U}_q$  for every  $q \geq 0$ . Thus  $\beta \in \bigcap_{q=0}^\infty \overline{U}_q = U_\infty$ . We have therefore obtained that  $U_\infty \cap \sigma^l(U_\infty) \neq \emptyset$  as this set contains  $\beta$ . This contradicts (U4A) and finishes the proof of (6.5). So, letting  $n \geq N_k(M)$  and  $\omega \in U_n$ , we get

$$(6.7) \quad \begin{aligned} \mathcal{L}_n^k \mathbb{1}(\omega) &= \mathcal{L}^k(\mathbb{1}_n^1)(\omega) \\ &= \sum_{\tau \in \sigma_c^{-k}(\omega)} \mathbb{1}_n^1(\tau) e^{\varphi_k(\tau)} + \sum_{\tau \in \sigma_0^{-k}(\omega)} \mathbb{1}_n^1(\tau) e^{\varphi_k(\tau)} \\ &= \sum_{\tau \in \sigma_c^{-k}(\omega)} e^{\varphi_k(\tau)} = \mathcal{L}^k \mathbb{1}(\omega) - \sum_{\tau \in \sigma_0^{-k}(\omega)} e^{\varphi_k(\tau)} \\ &= \mathbb{1}(\omega) - \sum_{\tau \in \sigma_0^{-k}(\omega)} e^{\varphi_k(\tau)}. \end{aligned}$$



Now, if  $\tau \in \sigma_0^{-k}(\omega)$ , then  $\gamma := \sigma^{j_\tau}(\tau) \in U_n$  with some  $0 \leq j_\tau \leq k-1$ , and using (6.5), we get

$$\begin{aligned}
 S_0(\omega) &:= \sum_{\tau \in \sigma_0^{-k}(\omega)} e^{\varphi_k(\tau)} \preceq \sum_{\tau \in \sigma_0^{-k}(\omega)} \mu_\varphi([\tau]) = \mu_\varphi \left( \sum_{\tau \in \sigma_0^{-k}(\omega)} [\tau] \right) \\
 &\leq \mu_\varphi \left( \bigcup_{j=0}^{k-1} \sigma^{-j} \left( \bigcup_{e \geq M} [e] \right) \right) \\
 (6.8) \quad &\leq \sum_{j=0}^{k-1} \mu_\varphi \left( \sigma^{-j} \left( \bigcup_{e \geq M} [e] \right) \right) = \sum_{j=0}^{k-1} \mu_\varphi \left( \bigcup_{e \geq M} [e] \right) \\
 &= k \mu_\varphi \left( \bigcup_{e \geq M} [e] \right).
 \end{aligned}$$

This means that there exists a constant  $C > 0$  such that

$$S_0(\omega) \leq C k \mu_\varphi \left( \bigcup_{e \geq M} [e] \right).$$

Denote the number  $C \mu_\varphi \left( \bigcup_{e \geq M} [e] \right)$  by  $\eta_M$ . Using (6.8), (6.7), and Proposition 5.2, we get the following.

$$\begin{aligned}
 \left| 1 - \int Q_n \mathbb{1} \, d\nu_n \right| &= \left| \int \mathbb{1} \, d\nu_n - \int Q_n \mathbb{1} \, d\nu_n \right| = \left| \int (\mathcal{L}_n^k \mathbb{1} + S_0) \, d\nu_n - \int Q_n \mathbb{1} \, d\nu_n \right| \\
 &= \left| \int (\mathcal{L}_n^k - \lambda_n^k Q_n) \mathbb{1} \, d\nu_n + \int (\lambda_n^k - 1) Q_n \mathbb{1} \, d\nu_n + \int S_0 \, d\nu_n \right| \\
 &\leq \int |\Delta_n^k \mathbb{1}| \, d\nu_n + |\lambda_n^k - 1| \cdot \|Q_n \mathbb{1}\|_\infty + \int S_0 \, d\nu_n \\
 &\leq C \kappa^n + C |\lambda_n^k - 1| + k \eta_M.
 \end{aligned}$$

Now, fix  $\varepsilon > 0$ . Take then  $n \geq 1$  so large that  $C \kappa^n < \varepsilon/3$ . Next, take  $M \geq 1$  so large that  $k \eta_M < \varepsilon/3$ . Finally take any  $n \geq N_k(M)$  so large that  $C |\lambda_n^k - 1| < \varepsilon/3$ . Then  $|1 - \int Q_n \mathbb{1} \, d\nu_n| < \varepsilon$ , and the proof is complete.  $\square$

The proof of the next lemma, corresponding to Lemma 4.3 in [8], goes through unaltered in the case of an infinite alphabet. We include it here for the sake of completeness and for the convenience of the reader.

**Lemma 6.3.** *If (U1)-(U4A) and (U5) hold, then*

$$\lim_{n \rightarrow \infty} \frac{\int Q_n (\mathcal{L} \mathbb{1}_n^c) \, d\nu_n}{\mu_\varphi(U_n)} = \lambda.$$

*Proof.* We assume without loss of generality that  $\lambda = 1$ . Let  $\tau_n : U_n \rightarrow U_n$  be the first return time from  $U_n$  to  $U_n$  under the shift map  $\sigma : E_A^\infty \rightarrow E_A^\infty$ . It is defined as

$$\tau_n(\omega) := \inf\{k \geq 1 : \sigma^k(\omega) \in U_n\}.$$

By Poincaré's Recurrence Theorem,  $\tau_n(\omega) < +\infty$  for  $\mu_\varphi$ -a.e.  $\omega \in E_A^\infty$ . We deal with the concept of first return time and first return time more thoroughly in Sections 14, 15, and 16. We have

$$\begin{aligned} \int_{U_n} \tau_n d\nu_n &= \sum_{i=1}^{\infty} i\nu_n(\tau_n^{-1}(i)) = \sum_{i=1}^{\infty} i\nu_n(\mathbb{1}_{\tau_n^{-1}(i)}) = \nu_n(\tau_n^{-1}(1)) + \sum_{i=2}^{\infty} i\nu_n(\mathbb{1}_n^{i-1} \circ \sigma \cdot \mathbb{1}_n^c \circ \sigma^i) \\ &= \nu_n(\tau_n^{-1}(1)) + \sum_{i=2}^{\infty} \frac{i}{\mu_\varphi(U_n)} \mu_\varphi(\mathbb{1}_n^{i-1} \circ \sigma \cdot \mathbb{1}_n^c \circ \sigma^i) \\ &= \nu_n(\tau_n^{-1}(1)) + \sum_{i=2}^{\infty} \frac{i}{\mu_\varphi(U_n)} \mu_\varphi(\mathcal{L}^i((\mathbb{1}_n^{i-1} \circ \sigma \cdot \mathbb{1}_n^c \circ \sigma^i))). \end{aligned}$$

Now using several times the property  $\mathcal{L}^j(f \cdot g \circ \sigma^j) = g\mathcal{L}^j(f)$ , a formal calculation leads to

$$\int_{U_n} \tau_n d\nu_n = \nu_n(\tau_n^{-1}(1)) + \sum_{i=2}^{\infty} i\nu_n(\mathcal{L}_n^{i-1}(\mathcal{L}(\mathbb{1}_n^c))).$$

Invoking at this point Proposition 5.2, we further get

$$\begin{aligned} \int_{U_n} \tau_n d\nu_n &= \nu_n(\tau_n^{-1}(1)) + \sum_{i=2}^{\infty} i\nu_n(\lambda_n^{i-1} Q_n \mathcal{L}(\mathbb{1}_n^c) + \Delta_n^{i-1} \mathcal{L}(\mathbb{1}_n^c)) \\ &= \nu_n(\tau_n^{-1}(1)) + \nu_n(Q_n \mathcal{L}(\mathbb{1}_n^c)) \sum_{i=2}^{\infty} i\lambda_n^{i-1} + \sum_{i=2}^{\infty} i\nu_n(\Delta_n^{i-1} \mathcal{L}(\mathbb{1}_n^c)) \\ &= \nu_n(\tau_n^{-1}(1)) + \nu_n(Q_n \mathcal{L}(\mathbb{1}_n^c)) \left( \frac{1}{(1-\lambda_n)^2} - 1 \right) + \sum_{i=2}^{\infty} i\nu_n(\Delta_n^{i-1}(\mathcal{L}\mathbb{1} - \mathcal{L}\mathbb{1}_n)) \\ &= \nu_n(\tau_n^{-1}(1)) + \nu_n(Q_n \mathcal{L}(\mathbb{1}_n^c)) \left( \frac{1}{(1-\lambda_n)^2} - 1 \right) + \sum_{i=2}^{\infty} i\nu_n(\Delta_n^{i-1}(\mathcal{L}\mathbb{1} - \mathcal{L}_n\mathbb{1})) \\ &= \nu_n(\tau_n^{-1}(1)) + \nu_n(Q_n \mathcal{L}(\mathbb{1}_n^c)) \left( \frac{1}{(1-\lambda_n)^2} - 1 \right) + \\ &\quad + \sum_{i=2}^{\infty} i\nu_n(\Delta_n^{i-1}(\mathcal{L}\mathbb{1})) - \sum_{i=2}^{\infty} i\nu_n(\Delta_n^i\mathbb{1}). \end{aligned}$$

Since, By Kac's Theorem,  $\int_{U_n} \tau_n d\nu_n = 1/\mu_\varphi(U_n)$ , multiplying both sides of this formula by  $\nu_n(Q_n \mathcal{L}(\mathbb{1}_n^c))$ , we thus get

$$\begin{aligned} \frac{\nu_n(Q_n \mathcal{L}(\mathbb{1}_n^c))}{\mu_\varphi(U_n)} &= \left( \frac{\nu_n(Q_n \mathcal{L}(\mathbb{1}_n^c))}{1 - \lambda_n} \right)^2 + \nu_n(Q_n \mathcal{L}(\mathbb{1}_n^c)) \left( \nu_n(\tau_n^{-1}(1)) - \nu_n(Q_n \mathcal{L}(\mathbb{1}_n^c)) + \right. \\ &\quad \left. + \sum_{i=2}^{\infty} i \nu_n(\Delta_n^{i-1}(\mathcal{L}\mathbb{1})) - \sum_{i=2}^{\infty} i \nu_n(\Delta_n \mathbb{1}) \right). \end{aligned}$$

Since, by Lemma 6.2,

$$\lim_{n \rightarrow \infty} \frac{\nu_n(Q_n \mathcal{L}(\mathbb{1}_n^c))}{1 - \lambda_n} = 1,$$

we have that  $\lim_{n \rightarrow \infty} \nu_n(Q_n \mathcal{L}(\mathbb{1}_n^c)) = 0$ , and since, applying Proposition 5.2 again, we deduce that the four terms in the big parentheses above are bounded, we get that

$$\lim_{n \rightarrow \infty} \frac{\nu_n(Q_n \mathcal{L}(\mathbb{1}_n^c))}{\mu_\varphi(U_n)} = 1.$$

The proof is complete. □

We shall prove the following.

**Lemma 6.4.** *If (U1)-(U4B) and (U5) hold, then*

$$\lim_{n \rightarrow \infty} \frac{\int Q_n(\mathcal{L}\mathbb{1}_n^c) d\nu_n}{\lambda - \lambda_n} = \lim_{n \rightarrow \infty} \int_{E_A^\infty} Q_n \mathbb{1} d\nu_n = 1 - \lambda^{-p} e^{\varphi_p(\xi)}.$$

*Proof.* Assume again without loss of generality that  $\mathcal{L}$  is normalized so that  $\lambda = 1$  and  $\mathcal{L}\mathbb{1} = \mathbb{1}$ . The first equality is general and has been established at the beginning of the proof of Lemma 6.2. We will thus concentrate on the second one. So, fix  $\omega \in U_n$  and  $k$ , an integral multiple of  $p$ , say  $k = qp$  with  $q \geq 0$ . Define the sets  $\sigma_0^{-k}(\omega)$  and  $\sigma_c^{-k}(\omega)$  exactly as in the proof of Lemma 6.2, i.e. by formulae (6.3) and (6.4). We further repeat the proof of Lemma 6.2 verbatim until formula (6.5), which now takes on the form:

$$\text{Either both } k - j \geq p \text{ and } \gamma|_{k-j} = \xi|_{k-j} \text{ or else } \sum_{i=1}^{k-j} \gamma_i \geq Mk.$$

Indeed, this is an immediate consequence of (U4B) and (U5). In other words

$$\sigma_0^{-k}(\omega) = \sigma_1^{-k}(\omega) \cup \sigma_2^{-k}(\omega),$$

where

$$\sigma_1^{-k}(\omega) := \left\{ \tau \in \sigma_0^{-k}(\omega) : \exists (0 \leq j \leq q-1) \sigma^{pj}(\tau) \in U_n \text{ and } \sigma^{pj}(\tau)|_{p(q-j)} = (\xi|_p)^{q-j} \right\}$$

and

$$\begin{aligned} \sigma_2^{-k}(\omega) &= \sigma_0^{-k}(\omega) \setminus \sigma_1^{-k}(\omega) \\ &\subseteq \left\{ \tau \in \sigma_0^{-k}(\omega) : \exists (0 \leq j \leq k-1) \sigma^j(\tau) \in U_n \text{ and } \sum_{i=j+1}^k \tau_i \geq Mk \right\}. \end{aligned}$$

Now, we shall prove that

$$(6.9) \quad \sigma_1^{-k}(\omega) = Z := \left\{ \tau \in \sigma_0^{-k}(\omega) : \sigma^{k-p}(\tau) \in [\xi|_p] \right\}.$$

Indeed, denote the set on the right-hand side of this equality by  $Z$ . If  $\tau \in Z$ , then  $\sigma^{p(q-1)}(\tau)|_p = \xi|_p$  and

$$\sigma^{p(q-1)}(\tau) = (\sigma^{p(q-1)}(\tau))|_p \sigma^{pq}(\tau) = \xi|_p \omega \in \xi|_p U_n \subseteq U_n,$$

where the last inclusion is due to (U4B). Thus, taking  $j = q-1$ , we see that  $\tau \in \sigma_1^{-k}(\omega)$ . So, the inclusion

$$(6.10) \quad Z \subseteq \sigma_1^{-k}(\omega)$$

has been established. In order to prove the opposite inclusion, let  $\tau \in \sigma_1^{-k}(\omega)$ . Then there exists  $j \in \{0, 1, \dots, q-1\}$  such that  $\sigma^{pj}(\tau) \in U_n$  and  $\sigma^{pj}(\tau)|_{p(q-j)} = (\xi|_p)^{q-j}$ . Then

$$\sigma^{k-p}(\tau)|_p = (\sigma^{p(q-j-1)} \circ \sigma^{pj}(\tau))|_p = \sigma^{pj}(\tau)|_{p(q-j-1)+1}^{p(q-j)+1} = \xi|_p,$$

and so,  $\tau \in Z$ . This establishes the inclusion  $\sigma_1^{-k}(\omega) \subseteq Z$ , and, together with (6.10) completes the proof of (6.9).

Therefore, keeping  $\omega \in U_n$  and using (6.9) and (6.7) we can write

$$\begin{aligned} \mathcal{L}_n^k(\omega) &= \mathcal{L}^k(\mathbb{1}_n^1)(\omega) \\ &= \mathbb{1}(\omega) - \sum_{\tau \in \sigma_1^{-k}(\omega)} e^{\varphi_k(\tau)} - \sum_{\tau \in \sigma_2^{-k}(\omega)} e^{\varphi_k(\tau)} \\ &= \mathbb{1}(\omega) - \sum_{\tau \in \sigma^{-k}(\omega)} \mathbb{1}_{[\xi|_p]} \circ \sigma^{p(q-1)}(\tau) e^{\varphi_k(\tau)} - \sum_{\tau \in \sigma_2^{-k}(\omega)} e^{\varphi_k(\tau)} \\ (6.11) \quad &= \mathbb{1}(\omega) - \mathcal{L}^{pq}(\mathbb{1}_{[\xi|_p]} \circ \sigma^{p(q-1)})(\omega) - \sum_{\tau \in \sigma_2^{-k}(\omega)} e^{\varphi_k(\tau)} \\ &= \mathbb{1}(\omega) - \mathcal{L}^p(\mathbb{1}_{[\xi|_p]})(\omega) - \sum_{\tau \in \sigma_2^{-k}(\omega)} e^{\varphi_k(\tau)}. \end{aligned}$$

Putting

$$S_2(\omega) := \sum_{\tau \in \sigma_2^{-k}(\omega)} e^{\varphi_k(\tau)}$$

and keeping  $\eta_M$  the same as in the proof of Lemma 6.2, the same estimates as in (6.8), give us

$$S_2(\omega) \leq k\eta_M.$$

Hence, using also (6.11), we get

$$\begin{aligned} \left| 1 - e^{\varphi_p(\xi)} - \int \mathcal{L}_n^k \mathbb{1} d\nu_n \right| &= \left| \int \mathcal{L}^p(\mathbb{1}_{[\xi|_p]}) d\nu_n - e^{\varphi_p(\xi)} + \int S_2 d\nu_n \right| = \\ &= \left| \int (e^{\varphi_p(\xi|_p\omega)} - e^{\varphi_p(\xi)}) d\nu_n + \int S_2 d\nu_n \right| \\ &\leq \int |e^{\varphi_p(\xi|_p\omega)} - e^{\varphi_p(\xi)}| d\nu_n + \int S_2 d\nu_n \\ &\leq \varepsilon_n + k\eta_M, \end{aligned}$$

with some  $\varepsilon_n \rightarrow 0$  resulting from the last item of (U4B). Hence, keeping  $k$  fixed and letting  $M$  and then  $n$  to infinity, we obtain

$$(6.12) \quad \lim_{n \rightarrow \infty} \int \mathcal{L}_n^k \mathbb{1} d\nu_n = 1 - e^{\varphi_p(\xi)}$$

for every  $k = qp \geq 1$ . Using Proposition 5.2, we get

$$\begin{aligned} \left| \int \mathcal{L}_n^k \mathbb{1} d\nu_n - \int Q_n \mathbb{1} d\nu_n \right| &= \left| \int (\mathcal{L}_n^k - \lambda_n^k Q_n) \mathbb{1} d\nu_n + \int (\lambda_n^k - 1) Q_n \mathbb{1} d\nu_n \right| \\ &\leq \left\| (\mathcal{L}_n^k - \lambda_n^k Q_n) \mathbb{1} \right\|_\infty + |\lambda_n^k - 1| \cdot \|Q_n \mathbb{1}\|_\infty \\ &\leq \|\Delta_n^k\|_\infty + C|\lambda_n^k - 1| \\ &\leq C\kappa^k + C|\lambda_n^k - 1|. \end{aligned}$$

So, fixing  $\varepsilon > 0$ , we first take and fix  $k \geq 1$  large enough so that  $C\kappa^k < \varepsilon/2$ , and then using Proposition 5.2, we take  $n \geq 1$  large enough so that  $C|\lambda_n^k - 1| < \varepsilon/2$ . Combining this with (6.12), we finally get the desired equality

$$\lim_{n \rightarrow \infty} \int Q_n \mathbb{1} d\nu_n = 1 - e^{\varphi^{(p)}(\xi)}.$$

The proof is complete. □

Applying Lemma 6.4 and proceeding along the lines of the proof of Lemma 6.3 (or Lemma 4.3 in [8]), we get the following analogue of Lemma 4.5 from [8].

**Lemma 6.5.** *If (U1)-(U4B) and (U5) hold, then*

$$\lim_{n \rightarrow \infty} \frac{\int Q_n(\mathcal{L}_n^c \mathbb{1}_n) d\nu_n}{\mu_\varphi(U_n)} = \lambda(1 - \lambda^{-p} e^{\varphi_p(\xi)})^2$$

Having proved Lemmas 6.2, 6.3, 6.4, and 6.5, Proposition 6.1 follows.

We now recall a basic escape rates definitions. Let  $G$  be an arbitrary subset of  $E_A^\infty$ . We set

$$(6.13) \quad \underline{R}_{\mu_\varphi}(G) := - \overline{\lim}_{k \rightarrow +\infty} \frac{1}{k} \log \mu_\varphi \left( \{ \omega \in E_A^\infty : \sigma^i(\omega) \notin G \text{ for all } i = 1, \dots, k \} \right)$$

and

$$(6.14) \quad \overline{R}_{\mu_\varphi}(G) := - \underline{\lim}_{k \rightarrow +\infty} \frac{1}{k} \log \mu_\varphi \left( \{ \omega \in E_A^\infty : \sigma^i(\omega) \notin G \text{ for all } i = 1, \dots, k \} \right).$$

We call  $\underline{R}_{\mu_\varphi}(G)$  and  $\overline{R}_{\mu_\varphi}(G)$  respectively the lower and the upper escape rate of  $G$ . Of course

$$\underline{R}_{\mu_\varphi}(G) \leq \overline{R}_{\mu_\varphi}(G),$$

and if these two numbers happen to be equal, we denote their common value by

$$R_{\mu_\varphi}(G)$$

and call it the escape rate of  $G$ . We provide here for the sake of completeness and convenience of the reader the short elegant proof, entirely taken from [8], of the following.

**Lemma 6.6.** *If (U0)-(U5) hold, then for all integers  $n \geq 0$  large enough the escape rates  $R_{\mu_\varphi}(U_n)$  exist, and moreover*

$$R_{\mu_\varphi}(U_n) = \log \lambda - \log \lambda_n.$$

*Proof.* Assume without loss of generality that the Perron-Frobenius operator  $\mathcal{L} : \mathcal{B}_\theta \rightarrow \mathcal{B}_\theta$  is fully normalized so that  $\lambda = 1$  and  $\mathcal{L}\mathbb{1} = \mathbb{1}$ . By virtue of Proposition 5.2 (b), (c), and (d), we have for every  $n \geq 0$  large enough and for all  $k \geq 1$  that

$$(6.15) \quad \begin{aligned} \mu_\varphi \left( \{ \omega \in E_A^\infty : \sigma^i(\omega) \notin U_n \text{ for all } i = 1, \dots, k \} \right) &= \\ &= \mu_\varphi \left( \bigcap_{j=0}^{k-1} \sigma^{-j}(U_n^c) \right) = \int_{E_A^\infty} \mathbb{1}_n^k d\mu_\varphi = \int_{E_A^\infty} \mathcal{L}^k(\mathbb{1}_n^k) d\mu_\varphi \\ &= \int_{E_A^\infty} \mathcal{L}_n^k(\mathbb{1}) d\mu_\varphi = \int_{E_A^\infty} (\lambda_n^k Q_n \mathbb{1} + \Delta_n^k \mathbb{1}) d\mu_\varphi \\ &= \lambda_n^k \int_{E_A^\infty} Q_n \mathbb{1} d\mu_\varphi + \int_{E_A^\infty} \Delta_n^k \mathbb{1} d\mu_\varphi. \end{aligned}$$

So, employing Proposition 5.2 (b) and Proposition 5.3, the latter to make sure that  $\lambda_n \in (0, +\infty)$  and  $\int_{E_A^\infty} Q_n \mathbb{1} d\mu_\varphi \in (0, +\infty)$ , we conclude from (6.15) with the help of Proposition 5.2 (e) and (g), that the limit

$$\lim_{k \rightarrow +\infty} \frac{1}{k} \log \mu_\varphi \left( \{ \omega \in E_A^\infty : \sigma^i(\omega) \notin U_n \text{ for all } i = 1, \dots, k \} \right)$$

exists and is equal to  $\log \lambda_n$ . The proof is complete.  $\square$

Now we are in position to prove the following main result of this section.

**Proposition 6.7.** *With the setting of Sections 3 and 4, assume that (U0)–(U5) hold. Then*

$$\lim_{n \rightarrow \infty} \frac{R_{\mu_\varphi}(U_n)}{\mu_\varphi(U_n)} = \begin{cases} 1 & \text{if (U4A) holds,} \\ 1 - \exp(\varphi_p(\xi) - pP(\varphi)) & \text{if (U4B) holds.} \end{cases}$$

*Proof.* By Lemma 6.6 we have

$$R_{\mu_\varphi}(U_n) = \frac{\log \lambda - \log \lambda_n}{\mu_\varphi(U_n)} = -\frac{\log \lambda_n - \log \lambda}{\lambda_n - \lambda} \cdot \frac{\lambda_n - \lambda}{\mu_\varphi(U_n)}.$$

Therefore, invoking Proposition 6.1, we get that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{R_{\mu_\varphi}(U_n)}{\mu_\varphi(U_n)} &= \lim_{n \rightarrow \infty} \frac{\log \lambda_n - \log \lambda}{\lambda_n - \lambda} \cdot \lim_{n \rightarrow \infty} \frac{\lambda_n - \lambda}{\mu_\varphi(U_n)} \\ &= \frac{1}{\lambda} \begin{cases} \lambda & \text{if (U4A) holds,} \\ \lambda(1 - \lambda^{-p} \exp(\varphi_p(\xi))) & \text{if (U4B) holds.} \end{cases} \\ &= \begin{cases} 1 & \text{if (U4A) holds,} \\ 1 - \exp(\varphi_p(\xi) - pP(\varphi)) & \text{if (U4B) holds.} \end{cases} \end{aligned}$$

The proof is complete.  $\square$

## Part 2. Escape Rates for Conformal GDMSs and IFSs

Our approach to proving results on escape rates for conformal graph directed Markov systems and conformal iterated function systems is based on the symbolic dynamics, more precisely, the symbolic thermodynamic formalism, developed in the preceding sections.

### 7. PRELIMINARIES ON CONFORMAL GDMSs

A Graph Directed Markov System (abbr. GDMS) consists of a directed multigraph and an associated incidence matrix,  $(V, E, i, t, A)$ . As earlier  $A$  is the incidence matrix, i. e.

$$A : E \times E \rightarrow \{0, 1\}$$

The multigraph consists of a finite set  $V$  of vertices and a countable (either finite or infinite) set of directed edges  $E$  and two functions  $i, t : E \rightarrow V$  together with a set of nonempty compact metric spaces  $\{X_v\}_{v \in V}$ , a number  $s$ ,  $0 < s < 1$ , and for every  $e \in E$ , a 1-to-1 contraction  $\varphi_e : X_{t(e)} \rightarrow X_{i(e)}$  with a Lipschitz constant  $\leq s$ . For brevity, the set

$$S = \{\varphi_e : X_{t(e)} \rightarrow X_{i(e)}\}_{e \in E}$$

is called a Graph Directed Markov System (abbr. GDMS). The main object of interest in this book will be the limit set of the system  $S$  and objects associated to this set. We now describe the limit set. For each  $\omega \in E_A^*$ , say  $\omega \in E_A^n$ , we consider the map coded by  $\omega$ :

$$\varphi_\omega = \varphi_{\omega_1} \circ \dots \circ \varphi_{\omega_n} : X_{t(\omega_n)} \rightarrow X_{i(\omega_1)}.$$

For  $\omega \in E_A^\infty$ , the sets  $\{\varphi_{\omega|n}(X_{t(\omega_n)})\}_{n \geq 1}$  form a descending sequence of non-empty compact sets and therefore  $\bigcap_{n \geq 1} \varphi_{\omega|n}(X_{t(\omega_n)}) \neq \emptyset$ . Since for every  $n \geq 1$ ,  $\text{diam}(\varphi_{\omega|n}(X_{t(\omega_n)})) \leq s^n \text{diam}(X_{t(\omega_n)}) \leq s^n \max\{\text{diam}(X_v) : v \in V\}$ , we conclude that the intersection

$$\bigcap_{n \geq 1} \varphi_{\omega|n}(X_{t(\omega_n)})$$

is a singleton and we denote its only element by  $\pi(\omega)$ . In this way we have defined the map

$$\pi : E_A^\infty \longrightarrow X := \bigoplus_{v \in V} X_v$$

from  $E_A^\infty$  to  $\bigoplus_{v \in V} X_v$ , the disjoint union of the compact sets  $X_v$ . The set

$$J = J_S = \pi(E_A^\infty)$$

will be called the limit set of the GDMS  $S$ .

In order to pass to geometry, we call a GDMS conformal (CGDMS) if the following conditions are satisfied.

(a) For every vertex  $v \in V$ ,  $X_v$  is a compact connected subset of a euclidean space  $\mathbb{R}^d$  (the dimension  $d$  common for all  $v \in V$ ) and  $X_v = \overline{\text{Int}(X_v)}$ .

(b) (Strong Open Set Condition) This consists of two parts:

(b1) (Open Set Condition) For all  $a, b \in E$ ,  $a \neq b$ ,

$$\varphi_a(\text{Int}(X_{t(a)})) \cap \varphi_b(\text{Int}(X_{t(b)})) = \emptyset,$$

and

(b2)

$$J_S \cap \text{Int}(X) = \emptyset.$$

(c) For every vertex  $v \in V$  there exists an open connected set  $W_v \supseteq X_v$  (where  $X = \bigcup_{v \in V} X_v$ ) such that for every  $e \in I$  with  $t(e) = v$ , the map  $\varphi_e$  extends to a  $C^1$  conformal diffeomorphism of  $W_v$  into  $W_{i(e)}$ .

(d) There are two constants  $L \geq 1$  and  $\alpha > 0$  such that

$$||\varphi'_e(y)| - |\varphi'_e(x)|| \leq L ||(\varphi'_e)^{-1}||^{-1} ||y - x||^\alpha.$$

for every  $e \in E$  and every pair of points  $x, y \in X_{t(e)}$ , where  $|\varphi'_\omega(x)|$  means the norm of the derivative.



**Remark 7.1.** If  $d \geq 2$  and a family  $S = \{\varphi_e\}_{e \in E}$  satisfies the conditions (a) and (c), then, due to Koebe's Distortion Theorem in dimension  $d = 2$  and the Loiuville Representation Theorem (stating that if  $d \geq 3$  then each conformal map is necessarily a Möbius transformation) it also satisfies condition (d) with  $\alpha = 1$ .

**Remark 7.2.** In the papers [13] and [14] there appeared also the so called Cone Condition. This condition was however exclusively needed only to prove the following.

**Theorem 7.3.** *If  $\mu$  is a Borel shift-invariant ergodic probability measure on  $E_A^\infty$ , then*

$$(7.1) \quad \mu \circ \pi^{-1}(\varphi_\omega(X_{t(\omega)}) \cap \varphi_\tau(X_{t(\tau)})) = 0$$

for all incomparable words  $\omega, \tau \in E^*$ .

This theorem is of particular importance if measure  $\mu$  is a Gibbs state of a Hölder continuous function. The following slight strengthening of Theorem 7.3 however immediately follows from the Strong Open Set Condition.

**Theorem 7.4.** *If  $\mu$  is a Borel shift-invariant ergodic probability measure on  $E_A^\infty$  with full topological support, then*

$$(7.2) \quad \mu \circ \pi^{-1}(\varphi_\omega(X_{t(\omega)}) \cap \varphi_\tau(X_{t(\tau)})) = 0$$

for all incomparable words  $\omega, \tau \in E^*$ .

Indeed, the Strong Open Set Condition ensures that for such measures  $\mu$

$$\mu(\text{Int}(X)) > 0$$

and, since we clearly have,

$$\sigma^{-1}(\pi^{-1}(\text{Int}(X))) \subseteq \text{Int}(X),$$

we thus conclude from ergodicity of  $\mu$  that  $\mu(\text{Int}(X)) = 0$ . The assertion of Theorem 7.4 thus follows. Note also that all Gibbs states are of full support.

We would like however to complete this comment by saying that in the case of finite alphabet  $E$  the Open set Condition alone suffices, and the item (b2) is not needed at all. It is not needed in the case of infinite alphabet either as long as we are only interested in the Hausdorff dimension of the limit set, i. e. as long as we only want prove Bowen's Formula.

Let  $F = \{f^{(e)} : X_{t(e)} \rightarrow \mathbb{R} : e \in E\}$  be a family of real-valued functions. For every  $n \geq 1$  and  $\beta > 0$  let

$$V_n(F) = \sup_{\omega \in E^n} \sup_{x, y \in X_{t(\omega)}} \{|f^{(\omega_1)}(\varphi_{\sigma(\omega)}(x)) - f^{(\omega_1)}(\varphi_{\sigma(\omega)}(y))|\} \varepsilon^{\beta(n-1)},$$

We have made the conventions that the empty word  $\emptyset$  is the only word of length 0 and  $\varphi_\emptyset = \text{Id}_X$ . Thus,  $V_1(F) < \infty$  simply means the diameters of the sets  $f^{(e)}(X)$  are uniformly bounded. The collection  $F$  is called a Hölder family of functions (of order  $\beta$ ) if

$$(7.3) \quad V_\beta(F) = \sup_{n \geq 1} \{V_n(F)\} < \infty.$$

We call the Hölder family  $F$ , summable (of order  $\beta$ ) if (7.3) is satisfied and

$$(7.4) \quad \sum_{e \in E} \exp \left( \sup (f|_{[e]}) \right) < +\infty.$$

In order to get the link with the previous sections on thermodynamic formalism on symbol spaces, we introduce now a potential function or amalgamated function,  $f : E^\infty \rightarrow \mathbb{R}$ , induced by the family of functions  $F$  as follows.

$$f(\omega) = f^{(\omega_1)}(\pi(\sigma(\omega))).$$

Our convention will be to use lower case letters for the potential function corresponding to a given Hölder system of functions. The following lemma is a straightforward, see [14] for a proof.

**Lemma 7.5.** *If  $F$  is a Hölder family (of order  $\beta$ ) then the amalgamated function  $f$  is Hölder continuous (of order  $\beta$ ). If  $F$  is summable, then so is  $f$ .*

We recall from [13] and [14] the following definitions:

$$\theta_{\mathcal{S}} := \inf \Gamma_{\mathcal{S}} \text{ where } \Gamma_{\mathcal{S}} = \inf \left\{ s \geq 0 : \sum_{e \in E} \|\varphi'_e\|_\infty^s < +\infty \right\}.$$

The proofs of the following two statements can be found in [14].

**Proposition 7.6.** *If  $\mathcal{S}$  is an irreducible conformal GDMS, then for every  $s \geq 0$  we have that*

$$\Gamma_{\mathcal{S}} = \{s \geq 0 : P(s) < +\infty\}.$$

*In particular,*

$$\theta_{\mathcal{S}} := \inf \{s \geq 0 : P(s) < +\infty\}.$$

**Theorem 7.7.** *If  $\mathcal{S}$  is a finitely irreducible conformal GDMS, then the function  $\Gamma_{\mathcal{S}} \ni s \mapsto P(s)$  is*

- *strictly decreasing,*
- *real-analytic,*
- *convex, and*
- $\lim_{s \rightarrow +\infty} P(s) = -\infty.$

We also introduce the following important characteristic of the system  $\mathcal{S}$ .

$$b_{\mathcal{S}} := \inf \{s \geq 0 : P(s) \leq 0\} \geq \theta_{\mathcal{S}}.$$

We call  $b_{\mathcal{S}}$  the Bowen's parameter of the system  $\mathcal{S}$ . The following theorem, providing a geometrical interpretation of this parameter has been proved in [14].

**Theorem 7.8.** *If  $\mathcal{S}$  is an finitely irreducible conformal GDMS, then*

$$\text{HD}(J_{\mathcal{S}}) = b_{\mathcal{S}} \geq \theta_{\mathcal{S}}.$$

Following [13] and [14] we call the system  $\mathcal{S}$  regular if there exists  $s \in (0, +\infty)$  such that

$$P(s) = 0.$$

Then by Theorems 7.8 and 7.7, such a zero is unique and is equal to  $b_{\mathcal{S}}$ .

We call the system  $\mathcal{S}$  strongly regular if there exists  $s \in [0, +\infty)$  (in fact in  $(\gamma_{\mathcal{S}}, +\infty)$ ) such that

$$0 < P(s) < +\infty.$$

By Theorem 7.7 each strongly regular conformal GDMS is regular.

Let  $\zeta : E_A^\infty \rightarrow \mathbb{R}$  be defined by the formula

$$(7.5) \quad \zeta(\omega) = \log |\varphi'_{\omega_1}(\pi(\sigma(\omega)))|.$$

Let us record the following obvious observation.

**Observation 7.9.** For every  $t \geq 0$ ,  $t\zeta$  is the amalgamated function of the following family of functions:

$$t\Xi := \{X_{t(e)} \ni x \mapsto t \log |\varphi'_e(x)| \in \mathbb{R}\}_{e \in E}.$$

The following proposition is easy to prove; see [14, Proposition 3.1.4] for complete details.

**Proposition 7.10.** *For every real  $t \geq 0$  the function  $t\zeta : E_A^\infty \rightarrow \mathbb{R}$  is Hölder continuous and  $t\Xi$  is a Hölder continuous family of functions.*

**Observation 7.11.** For every  $t \geq 0$  we have that  $t \in \Gamma_{\mathcal{S}}$  if and only if the Hölder continuous potential  $t\zeta$  is summable if and only if the Hölder continuous family of functions  $t\Xi$  is summable.

We denote:

$$P(\sigma, t\zeta) := P(t).$$

for every  $t \geq 0$ .

## 8. MORE TECHNICALITIES ON CONFORMAL GDMSs

We keep the setting and notation from the previous section.

- We call a point  $z \in X$  pseudo-periodic for  $\mathcal{S}$  if there exists  $\omega \in E_A^*$  such that  $z \in X_{t(\omega_0)}$  and  $\varphi_\omega(z) = z$ .
- We call a point  $z \in \mathcal{S}$  periodic for  $\mathcal{S}$  if  $z = \pi(\omega)$  for some periodic element  $\omega \in E_A^\infty$ .
- Of course every periodic point is pseudo-periodic. Also obviously, for maximal graph directed Markov systems, in particular for conformal iterated function systems, periodic points and pseudo-periodic points coincide.
- We call a periodic point  $z \in J_{\mathcal{S}}$  uniquely periodic if  $\pi^{-1}(z)$  is a singleton and there is exactly one  $\xi \in E_A^*$  such that the infinite concatenation  $\xi^\infty \in E_A^\infty$ ,  $\varphi_\xi(z) = z$ , and if  $\varphi_\alpha(z) = z$  for some  $\alpha \in E_A^*$ , then  $\alpha = \xi^q$  for some integer  $q \geq 1$ .

We shall prove the following.

**Lemma 8.1.** *If  $z \in J_{\mathcal{S}}$  is not pseudo-periodic for  $\mathcal{S}$ , then*

$$\pi^{-1}(z) \cap \bigcup_{k=1} \sigma^k(\pi^{-1}(z)) = \emptyset.$$

*Proof.* Assume for a contradiction that there exists  $\omega \in \pi^{-1}(z)$  such that  $\sigma^n(\omega) \in \pi^{-1}(z)$  for some  $n \geq 1$ . We then have

$$\varphi_{\omega|_n}(z) = \varphi_{\omega|_n}(\pi(\sigma^n(\omega))) = \pi(\omega|_n \sigma^n(\omega)) = \pi(\omega) = z.$$

So,  $z$  is pseudo-periodic, and this contradiction finishes the proof.  $\square$

In fact, we will need more:

**Lemma 8.2.** *Assume that  $z \in J_{\mathcal{S}}$  is not pseudo-periodic for the system  $\mathcal{S}$ . If  $k \geq 1$  is an integer,  $(l_n)_{n=1}^{\infty}$  is a sequence of integers in  $\{1, 2, \dots, k\}$ , and  $(\tau^{(n)})_{n=1}^{\infty}$  is a sequence of points in  $E_A^{\infty}$  such that*

$$\lim_{n \rightarrow \infty} \pi(\tau^{(n)}) = \lim_{n \rightarrow \infty} \pi(\sigma^{l_n}(\tau^{(n)})) = z,$$

*then*

$$\lim_{n \rightarrow \infty} \sum_{i=0}^{l_n} \tau_i^{(n)} = +\infty.$$

*Proof.* Seeking a contradiction suppose that

$$\varliminf_{n \rightarrow \infty} \sum_{i=0}^{l_n} \tau_i^{(n)} < +\infty.$$

Passing to a subsequence, we may assume without loss of generality

$$\varlimsup_{n \rightarrow \infty} \sum_{i=0}^{l_n} \tau_i^{(n)} < +\infty.$$

There then exists  $M \in (0, +\infty)$  such that

$$\sum_{i=0}^{l_n} \tau_i^{(n)} \leq M$$

for all  $n \geq 1$ . Hence,

$$\tau_i^{(n)} \leq M$$

for all  $n \geq 1$  and all  $i = 0, 1, 2, \dots, l_n$ . So, passing to yet another subsequence, we may further assume that the sequence  $(l_n)_{n=1}^{\infty}$  is constant, say  $l_n = l$  for all  $n \geq 1$ , and that for every  $i = 0, 1, 2, \dots, l$  the sequence  $(\tau_i^{(n)})_{n=1}^{\infty}$  is constant, say  $\tau_i^{(n)} = \tau_i \leq M$  for all  $n \geq 1$ . Let

$$\tau := \tau_1 \tau_2 \dots \tau_l.$$

it then follows from our hypothesis that

$$\begin{aligned}\varphi_\tau(z) &= \varphi_\tau\left(\lim_{n \rightarrow \infty} \pi(\sigma^l(\tau^{(n)}))\right) = \lim_{n \rightarrow \infty} \varphi(\tau(\pi(\sigma^l(\tau^{(n)})))) \\ &= \lim_{n \rightarrow \infty} \pi(\sigma^l(\tau^{(n)})) \\ &= \lim_{n \rightarrow \infty} \pi(\tau^{(n)}) = z.\end{aligned}$$

Thus  $z$  is a pseudo-periodic point for the graph directed Markov system  $\mathcal{S}$ , and this contradiction finishes the proof.  $\square$

A statement corresponding to Lemma 8.2 in the case of periodic points is the following.

**Lemma 8.3.** *Assume that  $z \in J_{\mathcal{S}}$  is uniquely periodic for the system  $\mathcal{S}$  (i.e.,  $\pi^{-1}(z)$  is a singleton and that there exists a unique point  $\xi \in E_A^*$  such that  $\xi^\infty \in E_A^\infty$ ,  $\varphi_\xi(z) = z$ , and if  $\varphi_\alpha(z) = z$  for some  $\alpha \in E_A^*$ , then  $\alpha = \xi^q$  for some integer  $q \geq 1$ ). Then if  $k \geq 1$  is an integer,  $(l_n)_{n=1}^\infty$  is a sequence of integers in  $\{1, 2, \dots, k\}$ , and  $(\tau^{(n)})_{n=1}^\infty$  is a sequence of points in  $E_A^\infty$  such that*

(a)

$$\lim_{n \rightarrow \infty} \pi(\tau^{(n)}) = \lim_{n \rightarrow \infty} \pi(\sigma^{l_n}(\tau^{(n)})) = z,$$

and

(b)

$$\overline{\lim}_{n \rightarrow \infty} \sum_{i=0}^{l_n} \tau_i^{(n)} < +\infty,$$

then  $l_n$  is an integral multiple of  $|\xi|$ , say  $l_n = q_n |\xi|$ , and

$$\tau^{(n)}|_{l_n} = \xi^{q_n}$$

for all  $n \geq 1$  large enough.

*Proof.* It follows from item (b) that there exists  $M \geq 1$  such that  $\tau_i^{(n)} \leq M$  for all  $n \geq 1$  and all  $1 \leq i \leq l_n$ . Assuming the contrapositive statement to our claim and passing to a subsequence, we may assume without loss of generality that the sequence  $(l_n)_{n=1}^\infty$  is constant, say  $l_n = l$  for all  $n \geq 1$ , and we may further assume that for every  $1 \leq i \leq l_n$  the sequence  $(\tau_i^{(n)})_{n=1}^\infty$  is constant, say  $\tau_i^{(n)} = \tau_i \in \{1, 2, \dots, M\}$  for all  $n \geq 1$  and

$$(8.1) \quad \tau_j^{(n)} \neq \xi_j$$

for all  $n \geq 1$  and some  $1 \leq j \leq l$ . Let

$$\tau := \tau_1 \tau_2 \dots \tau_l.$$

We now conclude, in exactly the same way as in the proof of Lemma 8.2 that  $\varphi_\tau(z) = z$ . Therefore, since  $z$  is uniquely pseudo-periodic, we get  $\tau = \xi^q$  with some  $q \geq 1$ . In particular  $q|\xi| = l$ , and so, using (8.1), we deduce that  $\tau \neq \xi^q$ . This contradiction finishes the proof.  $\square$

## 9. WEAKLY BOUNDARY THIN (WBT) MEASURES AND CONFORMAL GDMSs

In this section we first introduce the concept of Weakly Bounded Thin (WBT) measures. Roughly speaking, this notion relates the measure of an annulus to the measure of the ball it encloses. We prove some basic properties of (WBT) and provide some sufficient conditions for (WBT) to hold for a large class of measures on the limit set of a CGDMS. We were able to establish these properties, mainly due to the progress achieved in [19],

Let  $\mu$  be a Borel probability measure on a separable metric space  $(X, d)$ . For all  $\beta > 0$ ,  $x \in X$  and  $r > 0$ , let

$$A_\mu^\beta(x, r) := A(x; r - \mu(B(x, r))^\beta, r + \mu(B(x, r))^\beta),$$

where, in general,

$$A(z; r, R) := B(z, R) \setminus B(z, r)$$

is the annulus centered at  $z$  with the inner radius  $r$  and the outer radius  $R$ . We say that the measure  $\mu$  is weakly boundary thin (WBT) (with exponent  $\beta$ ) at the point  $x$  if

$$\lim_{r \rightarrow 0} \frac{\mu(A_\mu^\beta(x, r))}{\mu(B(x, r))} = 0.$$

Given  $\alpha > 0$ , we further define:

$$A_\mu^{\beta, \alpha}(x, r) := A(x; r - \alpha\mu(B(x, r))^\beta, r + \alpha\mu(B(x, r))^\beta).$$

The following proposition is obvious.

**Proposition 9.1.** *If  $\mu$  is a Borel probability measure on a separable metric space  $X$ , then for every point  $x \in \text{supp}(\mu)$ , the following are equivalent.*

- (a)  $\mu$  is (WBT) at  $x$ .
- (b) There exists  $\beta > 0$  such that the measure  $\mu$  is (WBT) at  $x$  with exponent  $\gamma > 0$  either if and only if  $\gamma \in (\beta, +\infty)$  or if and only if  $\gamma \in [\beta, +\infty)$ . Denote this  $\beta$  by  $\beta_\mu(x)$ .
- (c) There exist  $\alpha, \beta > 0$  such that

$$\lim_{r \rightarrow 0} \frac{\mu(A_\mu^{\beta, \alpha}(x, r))}{\mu(B(x, r))} = 0.$$

- (d) For every  $\beta \in (\beta_\mu(x), +\infty)$ ,

We say that a measure is weakly boundary thin (WBT) if it is (WBT) at every point of its topological support. We also say that a measure is weakly boundary thin almost everywhere (WBTAE) if it is (WBT) at almost every point. Of course (WBT) implies (WBTAE).

Now we aim to provide sufficient conditions for a Borel probability measure to be (WBT) and (WBTAE). Let  $\mu$  be an arbitrary Borel probability measure on a separable metric space.

Let  $\alpha > 0$ . We say that  $\mu$  is  $\alpha$ -upper Ahlfors ( $\alpha$ -up) at a point  $x \in X$  if there exists a constant  $C > 0$  (which may depend on  $x$ ) such that

$$\mu(B(x, r)) \leq Cr^\alpha$$

for all radii  $r > 0$ . Equivalently, for all radii  $r > 0$  sufficiently small. Following [23], the measure  $\mu$  is said to have the Thin Annuli Property (TAP) at a point  $x \in X$  if there exists  $\kappa > 0$  (which may depend on  $x$ ) such that

$$\lim_{r \rightarrow 0} \frac{\mu(A(x; r, r + r^\kappa))}{\mu(B(x, r))} = 0$$

We shall easily show the following.

**Proposition 9.2.** *Let  $(X, d)$  be a separable metric space, let  $\mu$  be a Borel probability measure on  $X$  and let  $\alpha > 0$ . If  $\mu$  is  $\alpha$ -upper Ahlfors with the Thin Annuli Property (TAP) at some  $x \in X$ , then  $\mu$  is (WBT) at  $x$ .*

*Proof.* Taking  $\beta > 0$  so large that  $C^\beta r^{\beta\alpha} \leq r^\kappa$  for  $r > 0$  small enough. Then for each radii  $r > 0$  we have that  $A_\mu^\beta(x; r, r + r^\kappa) \subset A(x, r, r + r^\kappa)$  and thus

$$0 \leq \limsup_{r \rightarrow 0} \frac{\mu(A_\mu^\beta(x, r))}{\mu(B(x, r))} \leq \limsup_{r \rightarrow 0} \frac{\mu(A(x, r, r + r^\kappa))}{\mu(B(x, r))} = 0$$

The proof is then complete. □

We recall from the book [23] that

$$\text{HD}_*(\mu) = \inf\{\text{HD}(Y) : Y \subset X \text{ is Borel and } \mu(Y) > 0\}.$$

We call  $\text{HD}_*(\mu)$  the lower Hausdorff dimension of  $\mu$ . The Hausdorff Dimension of  $\mu$  is commonly defined to be

$$\text{HD}(\mu) = \inf\{\text{HD}(Y) : Y \subset X \text{ is Borel and } \mu(Y) = 1\}$$

The reader should be aware that in [23] the above infimum is denoted  $\text{HD}^*(\mu)$  and is called the upper Hausdorff Dimension of  $\mu$ . We however, will use the more commonly accepted tradition rather than the point of view taken in [23]. Referring to the well-known fact (see [23] for instance) that if  $\mu(B(x, t)) \geq Cr^\gamma$  for the points  $x$  belonging to some Borel set  $F \subset X$  then  $\text{HD}(F) \leq \gamma$ , we immediately obtain the following.

**Lemma 9.3.** *If  $\text{HD}_*(\mu) > 0$  then  $\mu$  is  $\alpha$ -upper Ahlfors for every  $\alpha \in (0, \text{HD}_*(\mu))$  and  $\mu$ -a.e.  $x \in X$  with some constant  $C \in (0, +\infty)$  and every  $r > 0$  small enough.*

**Definition 9.4.** We say that a set  $J \subseteq \mathbb{R}^d$ ,  $d \geq 1$ , is geometrically irreducible if it is not contained in any countable union of conformal images of hyperplanes or spheres of dimension  $\leq d - 1$ .

**Observation 9.5.** Every set  $J \subseteq \mathbb{R}^d$ ,  $d \geq 1$ , with  $\text{HD}(J) > d - 1$  is geometrically irreducible.

**Observation 9.6.** If a set  $J \subseteq \mathbb{C}$  is not contained in any countable union of real analytic curves, then  $J$  is geometrically irreducible.

Now we can apply the results obtained above, in the context of CGMS. We shall prove the following.

**Theorem 9.7.** *Let  $\mathcal{S} = \{\varphi_e\}_{e \in E}$  be a finitely primitive CGDS satisfying the SOSC with a phase space  $X \subset \mathbb{R}^d$ . Let  $\psi : E_A^{\mathbb{N}} \rightarrow \mathbb{R}$  be a Hölder continuous strongly summable potential, the latter meaning that*

$$(9.1) \quad \sum_{e \in E} \exp(\inf(\varphi|_{[e]})) \|\varphi'_e\|^{-\beta} < +\infty$$

for some  $\beta > 0$ . As usual, let  $\mu_\psi$  denote its unique equilibrium state. If the limit set of  $J_{\mathcal{S}}$  is geometrically irreducible, then

- (a)  $\text{HD}_*(\mu_\psi \circ \pi_{\mathcal{S}}^{-1}) = \text{HD}(\mu_\psi \circ \pi_{\mathcal{S}}^{-1}) > 0$ ;
- (b) The measure  $\mu_\psi \circ \pi_{\mathcal{S}}^{-1}$  satisfies the Thin Annuli Property (TAP) at  $\mu_\psi \circ \pi_{\mathcal{S}}^{-1}$  a.e. point of  $\mathcal{S}$
- (c)  $\mu_\psi \circ \pi_{\mathcal{S}}^{-1}$  is (WBT) at  $\mu_\psi$  a.e. point of  $J_{\mathcal{S}}$ .

*Proof.* The proof of Theorem 4.4.2 in [13] [14] gives in fact that the measure  $\mu_\psi$  is dimensionally exact, i.e., that

$$\lim_{r \rightarrow 0} \frac{\log \mu_\psi \circ \pi_{\mathcal{S}}^{-1}(B(x, r))}{\log r}$$

exists for  $\mu_\psi \circ \pi_{\mathcal{S}}^{-1}$  for a.e.  $x \in J_{\mathcal{S}}$  and is equal to  $h_{\mu_\psi}(\sigma)/\chi_{\mu_\psi} > 0$ . A complete proof with all the details can be found in the last section of [4]. Therefore, property (a) is established. Property (b) follows now immediately from Theorem 30 in [19]. Condition (c) is now an immediate consequence of (a),(b), Lemma 9.3 and Proposition 9.2.  $\square$

**Remark 9.8.** Condition 9.1 is satisfied for instance for all potentials of the form  $E_A^{\mathbb{N}} \ni \omega \mapsto t \log |(\varphi'_{\mathcal{S}}(\pi_{\mathcal{S}}\sigma(\omega)))| \in \mathbb{R}$ , where  $t > \theta_{\mathcal{S}}$ . It also holds for  $t = \theta_{\mathcal{S}}$  if the system  $\mathcal{S}$  is strongly regular.

Now we shall deal with the case of a finite alphabet. We shall show that in the case of a finite alphabet (under a mild geometric condition in dimension  $d \geq 2$ ) the equilibrium states of all Hölder continuous potentials satisfy (WBT) at every point of the limit set. Thus our approach is complete in the case of the finite alphabet and present paper entirely covers the case of conformal IFSs (even GDMSs) with finite alphabet. We shall prove the following.

**Theorem 9.9.** *Let  $E$  be a finite set and let  $\mathcal{S} = \{\varphi_e\}_{e \in E}$  be a primitive conformal GDMS acting in the space  $\mathbb{R}$ . If  $\psi : E_A^{\infty} \rightarrow \mathbb{R}$  is an arbitrary Hölder continuous (with the phase space sets  $X_v \subset W_v \subset \mathbb{R}$ ,  $v \in V$ ) and  $\mu_\varphi$  is the corresponding equilibrium state on  $E_A^{\infty}$  then the projection measure  $\mu_\psi \circ \pi_{\mathcal{S}}^{-1}$  is (WBT) at every point of  $J_{\mathcal{S}}$ .*



*Proof.* Put

$$u := K^{-1} \min \{ \|\varphi'_e\| : e \in E \}$$

so that

$$|\varphi'_e(x)| \geq u$$

for all  $e \in E$  and all  $x \in X_{t(e)}$ . For ease of notation we denote

$$\widehat{\mu}_\psi := \mu_\psi \circ \pi_S^{-1}.$$

Fix  $\beta > 0$ . Consider the family

$$\mathcal{F}_\psi^\beta(z, r) := \{ \omega \in E_A^k : A_{\mu_\psi}^\beta(z, r) \cap \varphi_\omega(J_{|\omega|_1-1}) \neq \emptyset \text{ and } \|\varphi'_\omega\| \geq \mu_\psi(B(z, r)^\beta) \}$$

Now consider  $\widehat{\mathcal{F}}_\psi^\beta(z, r)$ , the family of all words in  $\mathcal{F}_\psi^\beta(z, r)$  that have no extensions to elements in  $\mathcal{F}_\psi^\beta(z, r)$ , where we don't consider a finite word to be an extension of itself. Note that then:

- (a)  $\widehat{\mathcal{F}}_\psi^\beta(z, r)$  consists of mutually incomparable words;
- (b)  $\bigcup_{\omega \in \widehat{\mathcal{F}}_\psi^\beta(z, r)} [w] \supset \pi_S^{-1}(A_{\mu_\psi}^\beta(z, r))$ ; and
- (c)  $\forall \omega \in \widehat{\mathcal{F}}_\psi^\beta(z, r), \mu_\psi(B(z, r)^\beta) \leq \|\varphi'_\omega\| \leq u^{-1} \mu_\psi(B(z, r)^\beta)$

Therefore the family

$$\{ \varphi_\omega(\text{Int}(X_{t(\omega)})) : \omega \in \widehat{\mathcal{F}}_\psi^\beta(z, r) \}$$

consists of mutually disjoint open sets each of which contains a ball of radius  $K^{-1} R \mu_\psi(B(z, r)^\beta)$  where  $R$  is as in the proof of Lemma 9.13. Since also

$$\bigcup_{\omega \in \widehat{\mathcal{F}}_\psi^\beta(z, r)} \varphi_\omega(X_{t(\omega)}) \subset A(z, r - (1 + DM^{-1})\mu_\psi(B(z, r)^\beta), r - (1 + DM^{-1})\mu_\psi(B(z, r)^\beta))$$

we obtain that

(9.2)

$$\begin{aligned} \#\widehat{\mathcal{F}}_\psi^\beta(z, r) &\leq \frac{\text{Leb}_1(A(z, r - (1 + DM^{-1})\mu_\psi(B(z, r)^\beta), r - (1 + Du^{-1})\mu_\psi(B(z, r)^\beta))}{2K^{-1} R \mu_\psi(B(z, r)^\beta)} \\ &\approx \frac{\mu_\psi^\beta(B(z, r))}{\mu_\psi^\beta(B(z, r))} = 1. \end{aligned}$$

So we have shown that the number of elements of  $\widehat{\mathcal{F}}_\psi^\beta(z, r)$  is uniformly bounded above, and in order to estimate  $\widehat{\mu}_\psi(A_{\mu_\psi}^\beta(z, r))$ . i.e. in order to complete the proof we now only need a sufficiently good upper bound on  $\mu_\psi([\omega])$  for all  $\omega \in \widehat{\mathcal{F}}_\psi^\beta(z, r)$ . We will do so now. It is well known (see [13], [14]) that there are two constants  $\eta \in (0, +\infty)$  and  $C \in (0, +\infty)$  such that

$$(9.3) \quad \mu_\psi([\tau]) \leq C \exp(-\eta(|\tau| + 1))$$

for all  $\tau \in E_A^*$ . Fix  $\omega \in \widehat{\mathcal{F}}_\psi^\beta(z, r)$ . Denote  $k := |\omega|$ . Invoking (c) we get that  $u^k \leq \|\varphi'_\omega\| \leq u^{-1} \mu_\psi^\beta(B(z, r))$ , whence

$$k + 1 \geq \frac{\beta \log \mu_\psi(B(z, r))}{\log u}.$$

Inserting this into (9.3) we get that

$$\mu_\psi([\omega]) \leq C \exp \left( -\beta \eta \frac{\log \mu_\psi(B(z, r))}{\log u} \right) = C \mu^{\gamma\beta}(B(z, r))$$

where  $\gamma = \frac{\eta}{\log(1/u)} \in (0, +\infty)$ . Having this and invoking (b) and (9.2) we obtain that

$$\frac{\widehat{\mu}_\psi(A_{\mu_\psi}^\beta(z, r))}{\mu_\psi(B(z, r))} \leq \frac{\widehat{\mu}_\psi(B(z, r))^{\gamma\beta}}{\widehat{\mu}_\psi(B(z, r))} \leq \mu_\psi(B(z, r))^{\gamma\beta-1}$$

and the proof is complete by noting that  $\lim_{r \rightarrow 0} \mu_\psi(B(z, r))^{\gamma\beta-1} = 0$  provide that we take  $\gamma > 1/\beta$ .  $\square$

Now we pass to the case of  $d \geq 2$ . We get the same full result as in the case of  $d = 1$  but with a small additional assumption that the conformal system  $\mathcal{S}$  is geometrically irreducible.

**Theorem 9.10.** *Let  $E$  be a finite set and let  $\mathcal{S} = \{\varphi_e\}_{e \in E}$  be a primitive geometrically irreducible conformal GDMS with the phase space sets  $X_v \subset W_v \subset \mathbb{R}^d$ . If  $\psi : E_A^\mathbb{N} \rightarrow \mathbb{R}$  is an arbitrary Hölder continuous potential and  $\mu_\varphi$  is the corresponding equilibrium state then the projection measure  $\mu_\psi \circ \pi_\mathcal{S}^{-1}$  is (WBT) at every point of  $J_\mathcal{S}$*

*Proof.* The meaning of  $\widehat{\mu}_\psi$  is exactly the same as in the proof of the previous theorem. The proof of the current theorem is entirely based on the following.

**Claim 1:** There are a constant  $\alpha > 0$  and  $C \in (0, +\infty)$  such that

$$\widehat{\mu}_\psi(B(\partial B(z, R), r)) \leq Cr^\alpha$$

for all  $z \in \mathbb{R}^d$  and all radii  $r, R > 0$ .

This claim is actually a sub-statement of formula (2.19) from [35] in a more specific setting. In particular:

- (a) [35] deals with finite IFSs rather than finite alphabet CGDMS;
- (b) [35] deals with Hölder continuous families of functions and their corresponding equilibrium states rather than Hölder continuous potentials on the symbol space  $E_A^\mathbb{N}$  and their projections; and
- (c) with the restrictions of (a) and (b) Claim 1 is a sub-statement of formula (2.19) from [35] only in the case of  $d \geq 3$ .

However, a close inspection of arguments leading to (2.19) of [35] indicates that the difference of (a) is inessential for these arguments, and for (b) that the only property of equilibrium states of Hölder continuous families of functions was that of being projections of Hölder continuous potentials from the symbol space  $E_A^\infty$ . Concerning (c) it only remains to consider the case  $d = 2$ . We then redefine the family  $\mathcal{F}_0$  from section 2, page 225 of [35] to conclude that also all the intersections of the form  $X \cap L$ , where  $L \subset \mathbb{C}$  where is a round circle (of arbitrary center and radius). The argument in [35] leading to (2.19) goes through with obvious minor modifications. Claim 1 is then established.

Using this claim, we obtain

$$\frac{\widehat{\mu}_\psi(A_{\mu_\psi}^\beta(z, r))}{\widehat{\mu}_\psi(B(x, r))} \leq \frac{C\widehat{\mu}_\psi^{\alpha\beta}(B(z, r))}{\widehat{\mu}_\psi(B(x, r))} = C\widehat{\mu}_\psi^{\alpha\beta-1}(B(z, r))$$

and the proof is complete and by noting that  $\lim_{r \rightarrow 0} \mu_\psi^{\alpha\beta-1}(B(z, r))$  for every  $\beta > 1/\alpha$ .  $\square$

Fixing a  $\kappa > 0$  let

$$N_\kappa(x, r) := \left\lceil -\frac{1}{\kappa} \log \mu(B(x, r)) \right\rceil \in \mathbb{N} \cup \{+\infty\}$$

where  $[t]$ ,  $t \in \mathbb{R}$ , denotes the integer part of  $t$ . Let us make right away an immediately evident, but extremely important, observation.

**Observation 9.11.** If  $\mu$  is a Borel probability measure on  $X$ , then for every  $r > 0$ , we have that

$$e^{-\kappa N_\kappa(x, r)} \leq \mu(B(x, r)) \leq e^\kappa e^{-\kappa N_\kappa(x, r)}.$$

Now, let in addition  $\mathcal{S} = \{\varphi_e\}_{e \in E}$  be a finitely primitive CGDMS with a phase space  $X \subseteq \mathbb{R}^d$ . For every  $x \in X$  and  $r > 0$  and an integer  $n \geq 0$ , let

$$A_n^*(x, r) := \bigcup \left\{ \varphi_\omega(J) : \omega \in E_A^n, \varphi_\omega(J) \cap B(x, r) \neq \emptyset \text{ and } \varphi_\omega(J) \cap B^c(x, r) \neq \emptyset \right\}.$$

We say that the measure  $\mu$  is dynamically boundary thin (DBT) at the point  $x \in \overline{J}_\mathcal{S}$  if for some  $\kappa > 0$

$$(9.4) \quad \lim_{r \rightarrow 0} \frac{\mu(A_{N_\kappa(x, r)}^*(x, r))}{\mu(B(x, r))} = 0.$$

We say that the measure  $\mu$  is Dynamically Boundary Thin (DBT) almost everywhere if the set of points where it fails to be (DBT) is measure zero, and that the measure  $\mu$  is Dynamically Boundary Thin (DBT) if it is (DBT) at every point of its topological support. We shall prove the following.

**Proposition 9.12.** *If a Borel probability measure  $\mu$  on  $\overline{J}_\mathcal{S}$  is (WBT) at some point  $x \in \overline{J}_\mathcal{S}$ , then it is (DBT) at  $x$ .*

*Proof.* Let  $\beta > 0$  be as in the definition of (WBT) of  $\mu$  at  $x$ . Since  $\mathcal{S}$  is a conformal GDMS, there exist constants  $\eta > 0$  and  $D \geq 1$  such that

$$\text{diam}(\varphi_\omega(X)) \leq D\eta^{-\eta|\omega|}$$

for all  $\omega \in E_A^*$ . Therefore, if  $\kappa > 0$  is sufficiently small, then for every  $x \in \overline{J}_\mathcal{S}$  and every  $r > 0$  we have

$$\begin{aligned} A_{N_\kappa(x,r)}^*(x, r) &\subseteq A(x; r - De^{-\kappa N_\kappa(x,r)}, r + De^{-\kappa N_\kappa(x,r)}) \\ &\subseteq A(x; r - D(\mu(B(x, r))^{\eta/\kappa}, r + D(\mu(B(x, r))^{\eta/\kappa}) \\ &= A_\mu^{\eta/\kappa, De}(x, r). \end{aligned}$$

For every  $r > 0$ , sufficiently small, we then have

$$\frac{\mu(A_{N_\kappa(x,r)}^*(x, r))}{\mu(B(x, r))} \leq \frac{\mu(A_\mu^{\eta/\kappa, De}(x, r))}{\mu(B(x, r))}.$$

Now, if  $\kappa > 0$  is sufficiently small, then  $\eta/\kappa > \beta$  and, in consequence,

$$0 \leq \lim_{r \rightarrow 0} \frac{\mu(A_{N_\kappa(x,r)}^*(x, r))}{\mu(B(x, r))} \leq \lim_{r \rightarrow 0} \frac{\mu(A_\mu^{\eta/\kappa, De}(x, r))}{\mu(B(x, r))} = 0.$$

This means that  $\mu$  is (DBT) at  $x$  and the proof is complete.  $\square$

Now we shall provide some sufficient conditions, different than (WBT), for (DBT) to hold at every point of  $J_\mathcal{S}$ . We will do it by developing the reasoning of Lemma 5.2 in [3]. We will not really make use of these conditions in the current manuscript but these are very close to the subject matter of the current section and will not occupy too much space. These may be needed in some future. We shall first prove the following.

**Lemma 9.13.** *Let  $\mathcal{S} = \{\varphi_e\}_{e \in E}$  be a finitely primitive CGDMS satisfying SOSC. Assume that a number  $t > \max\{\theta_\mathcal{S}, d - 1\}$  satisfies*

$$(9.5) \quad t > d - 1 + \frac{P(t)}{\log s}$$

*Denote by  $\mu_t$  the unique equilibrium state of the potential  $E_A^\infty \ni \omega \mapsto t \log |\varphi_{\omega_1}(\pi(\sigma\omega))|$ . Then there exists constants  $\alpha > 0$  and  $C \in [0, +\infty]$  such that*

$$\mu_t \circ \pi_\mathcal{S}^{-1}(A_k^*(z, r)) \leq Ce^{-\alpha k}$$

*for all  $z \in \overline{J}_\mathcal{S}$ , all radii  $r > 0$  and all integers  $n \geq 0$ .*

*Proof.* For all  $a \in E$ , let  $r > 0$ . Set

$$J_a := \bigcup_{b \in E: A_{ab}=1} \pi_\mathcal{S}([b]).$$

$r \in (0, 1]$ . For  $k \geq 0$  consider the set

$$E_A^k(z, r) := \left\{ \omega \in E_A^k : \varphi_\omega(J_{w_{|n|-1}}) \cap B(z, r) \neq \emptyset \text{ and } \varphi_\omega(J_{w_{|w|-1}}) \cap B(z, r)^c \neq \emptyset \right\}.$$

Furthermore, for every  $k \geq 0$  let

$$E_A^k(z, r; n) := \{\omega \in E_A^k(z, r) : s^{n+1} < \|\varphi'_w\| \leq s^n\}$$

Then the family

$$\mathcal{F}_k(z, r; n) := \{\varphi_\omega(\text{Int}(X)) : \omega \in E_A^k(z, r; n)\}$$

consists of mutually disjoint open sets contained in

$$A(z; r - Ds^n, r + Ds^n)$$

each of which contains a ball of radius  $K^{-1}Rs^{n+1}$  where  $R > 0$  is the radius of an open ball entirely contained in  $\text{Int}X_v$  for all  $v \in V$ . So, then

$$\#\mathcal{F}_k(z, r; n) \leq \frac{\text{Leb}_d(A(z; r - Ds^n, r + Ds^n))}{\text{Leb}_d(0, k^{-1}Rs^{n+1})} \leq C_1 \frac{r^{d-1}s^n}{s^{nd}} = C_1 r^{d-1} s^{(1-d)n} \leq C_1 s^{(1-d)n}$$

with the same universal constant  $C_1 \in (0, +\infty)$ . Since  $E_A^k(z, r, n) = \emptyset$  for every  $n < k$ , then knowing that  $t > \max\{\theta_S, d-1\}$  gives that

$$\begin{aligned} \mu_t \circ \pi_S^{-1}(A_k^*(z, v)) &= \sum_{n=k}^{\infty} \mu_t \left( \bigcup_{\omega \in E_A^k(z, r; k)} \varphi_\omega(\mathcal{J}_{|\omega|^{-1}}) \right) \\ &\leq \sum_{n=k}^{\infty} \#\mathcal{F}_k(z, r; n) \sup\{\mu_t(\varphi_\omega(X)) : \omega \in E_A(z, r; n)\} \\ &\leq C_1 \sum_{n=k}^{\infty} s^{(1-d)n} e^{-P(t)k} s^{tn} \\ &= C_1 e^{-P(t)k} \sum_{n=k}^{\infty} s^{(t+1-d)n} \\ &= C_1 (1 - s^{t+1-d})^{-1} e^{-P(t)k} s^{(t+d-1)k} \\ &= C_1 (1 - s^{t+d-1})^{-1} \exp((t+1-d) \log s - P(t)k) \end{aligned}$$

But  $(t+1-d) \log s - P(t) < 0$  by virtue of (9.5) and the proof is complete.  $\square$

As an immediate consequence of this lemma we get the following.

**Theorem 9.14.** *Let  $\mathcal{S} = \{\varphi_e\}_{e \in E}$  be a finitely primitive CGDMS satisfying SOSC. If a number  $t > \max\{\theta_S, d-1\}$  satisfies*

$$(9.6) \quad t > d-1 + \frac{P(t)}{\log s}$$

*then  $\mu_t \circ \pi_S^{-1}$ , the projection of the corresponding equilibrium state  $\mu_t$  on  $E_A^\infty$ , is DBT at every point of  $\overline{J_S}$*

*Proof.* Because of the Lemma 9.13 for all  $z \in \overline{J_S}$  and all radii  $r > 0$ , we have that

$$\begin{aligned} \mu_t \circ \pi_S^{-1} (A_{N_\kappa(z,r)}^*(z, r)) &\leq C \exp(-\alpha N_\kappa(z, r)) \\ &\leq C \exp \left( -\alpha \left( \frac{1}{\kappa} \log \mu_t \circ \pi_S^{-1}(B(z, r)) - 1 \right) \right) \\ &= C e^\alpha (\mu_t \circ \pi_S^{-1}(B(z, r)))^{\alpha/\kappa} \\ &= C e^\alpha (\mu_t \circ \pi_S^{-1}(B(z, r)))^{\frac{\alpha}{\kappa}-1} \mu_t \circ \pi_S^{-1}(B(z, r)) \end{aligned}$$

Equivalently,

$$\frac{\mu_t \circ \pi_S^{-1}(A_{N_\kappa(z,r)}^*(z, r))}{\mu_t \circ \pi_S^{-1}(B(z, r))} \leq C e^\alpha (\mu_t \circ \pi_S^{-1}(B(z, r)))^{\frac{\alpha}{\kappa}-1}$$

and the proof is complete since the right hand-side of this inequality converges to 0 as  $r \rightarrow 0$  for every  $\kappa \in (0, \alpha)$ .  $\square$

As an immediate consequence of this theorem we get the following.

**Corollary 9.15.** *Let  $\mathcal{S}$  be a finitely primitive strongly regular CGDMS satisfying SOSC. Then there exists  $\eta > 0$  such that if  $t \in (\max\{\theta_S, d-1\}, \text{HD}(J_S) + \eta)$ , then  $\mu_t \circ \pi_S^{-1}$ , the projection of the corresponding equilibrium state  $\mu_t$  on  $E_A^\infty$ , is DBT at every point of  $\overline{J_S}$ .*

*Proof.* We only need to check that if  $t \in (\max\{\theta_S, d-1\}, \text{HD}(J_S) + \eta)$  for some  $\eta > 0$  sufficiently small then (9.6) holds. Indeed, since  $P(b_S) = 0$  (by strong regularity of  $\mathcal{S}$ ) this is an immediate consequence of continuity of the function  $(\theta_S, +\infty) \ni t \mapsto P(t) \in \mathbb{R}$ .  $\square$

## 10. ESCAPE RATES FOR CONFORMAL GDMSs; MEASURES

In this section we continue the analysis from the previous section and we prove our first main results concerning escape rates; the one for conformal GDMSs and equilibrium/Gibbs measures. We first work for a while in full generality. Indeed, let  $\mu$  be an arbitrary Borel probability measure on a metric space  $(X, d)$ . Fix  $\kappa > 0$ . Fix  $z \in X$ . Let

$$\Gamma := \Gamma_\kappa(z) := \{N_\kappa(z, r) : 0 < r \leq 2\text{diam}(X)\}.$$

Represent  $\Gamma$  as a strictly increasing sequence  $(l_n)_{n=0}^\infty$  of non-negative integers. Let us record the following.

**Observation 10.1.** If  $z \in \text{supp}(\mu)$ , then  $\Gamma_\kappa(z) \subseteq \mathbb{N}$ . Moreover, the set  $\Gamma$  is infinite if and only if  $z$  is not an atom of  $\mu$ .

We shall prove the following.

**Lemma 10.2.** *If  $\mu$  is a Borel probability measure on  $X$  which is (WBT) at some point  $z \in X$ , then the set  $\Gamma_\kappa(z)$  has bounded gaps, precisely meaning that*

$$\Delta l(z) := \sup_{n \geq 0} \{l_{n+1} - l_n\} < +\infty$$

*Proof.* Fix  $n \geq 0$ . There then exists  $r_n > 0$  such that

$$N_\kappa(z, r) \leq l_n + 1$$

for all  $r > r_n$ , and

$$N_\kappa(z, r) \geq l_{n+1}$$

for all  $r < r_n$ . Therefore, by Observation 9.11

$$\mu(B(z, r_n)) \leq e^\kappa \exp(-\kappa l_{n+1})$$

and

$$\mu(B(z, r_n)) \geq \exp(-\kappa(l_n + 1)).$$

Hence,

$$\frac{\mu(B(z, r_n + \mu^\beta(B(z, r_n))))}{\mu(B(z, r_n))} \geq e^{-\kappa} \exp(\kappa(l_{n+1} - (l_n + 1))).$$

for all  $\beta > 0$ , in particular for  $\beta > \beta_\mu(z)$ . But since the measure  $\mu$  is (WBT) at  $z$ , we therefore have that

$$\overline{\lim}_{n \rightarrow \infty} \exp(\kappa(l_{n+1} - (l_n + 1))) \leq e^\kappa \lim_{n \rightarrow \infty} \frac{\mu(B(z, r_n + \mu^\beta(B(z, r_n))))}{\mu(B(z, r_n))} \leq e^\kappa.$$

Thus

$$\overline{\lim}_{n \rightarrow \infty} (l_{n+1} - (l_n + 1)) < +\infty$$

and the proof is complete.  $\square$

For every  $n \geq 0$  let

$$\mathcal{R}_n := \{r \in (0, 2\text{diam}(X)) : N_\kappa(z, r) = l_n\},$$

and, given in addition  $0 \leq m \leq n$ , let

$$\mathcal{R}(m, n) := \bigcup_{k=m}^n \mathcal{R}_k.$$

Now we make an additional substantial assumption that

$$\mathcal{S} = \{\varphi_e\}_{e \in E},$$

a conformal GDMS is given, and  $\text{supp}(\mu) = \overline{J}_\mathcal{S}$ . For any  $z \in J_\mathcal{S}$  and  $r \in (0, 2\text{diam}(X))$ , define

$$(10.1) \quad W^-(z, r) := B_{J_\mathcal{S}}(z, r) \setminus A_{N_\kappa(z, r)}^*(z, r) \quad \text{and} \quad W^+(z, r) := B_{J_\mathcal{S}}(z, r) \cup A_{N_\kappa(z, r)}^*(z, r).$$

Let us record the following two immediate consequences of this definition.

**Observation 10.3.** For every  $z \in J_\mathcal{S}$  and  $r \in (0, 2\text{diam}(X))$ , we have

$$W^-(z, r) \subseteq B(z, r) \subseteq W^+(z, r).$$

**Observation 10.4.** For every  $z \in J_\mathcal{S}$  and  $r \in (0, 2\text{diam}(X))$  both sets  $W^-(z, r)$  and  $W^+(z, r)$  can be represented as unions of cylinders of length  $N_\kappa(z, r)$ .

Fix  $\kappa > 0$  so small that (9.4) holds and so that  $\eta/\kappa > \beta_\mu(z)$ . We shall prove the following.

**Lemma 10.5.** *For all  $k \geq 0$  large enough, if  $n - k \geq 2$ , then*

$$W^+(z, s) \subseteq W^-(z, r)$$

for all  $r \in \mathcal{R}_k$  and all  $s \in \mathcal{R}_n$ .

*Proof.* The assertion of our lemma is equivalent to the statement that

$$W^+(z, s) \cap A_{l_k}^*(z, r) = \emptyset.$$

Assume for a contradiction that there are sequences  $(n_j)_{j=0}^\infty$  and  $(k_j)_{j=0}^\infty$  of positive integers such that  $\lim_{j \rightarrow \infty} k_j = +\infty$  and  $n_j - k_j \geq 2$  for all  $j \geq 0$ , and also there are radii  $r_j \in \mathcal{R}_{k_j}$  and  $s_j \in \mathcal{R}_{n_j}$  such that

$$W^+(z, s_j) \cap A_{l_{k_j}}^*(z, r_j) = \emptyset,$$

for all  $j \geq 0$ . Since we know that for each  $\omega \in E_A^*$ ,

$$\text{diam}(\varphi_\omega(J)) \leq D e^{-\eta|\omega|},$$

using Observation 9.11, we therefore conclude that

$$\begin{aligned} s_j + D\mu^{\eta/\kappa}(B(z, r_j)) &\geq s_j + D \exp(-\eta N_\kappa(z, r_j)) \geq s_j + D\varepsilon^{-\eta l_{k_j}} \\ &\geq s_j + D\varepsilon^{-\eta l_{n_j}} \geq r_j - D\varepsilon^{-\eta l_{k_j}} \\ &= r_j - D \exp(-\eta N_\kappa(z, r_j)) \\ &\geq r_j - D\mu^{\eta/\kappa}(B(z, r_j)). \end{aligned}$$

Hence,  $s_j \geq r_j - 2D\mu^{\eta/\kappa}(B(z, r_j))$ , and therefore,

$$(10.2) \quad \frac{\mu(B(z, s_j))}{\mu(B(z, r_j))} \geq \frac{\mu(B(z, r_j)) - \mu(A_\mu^{\eta/\kappa, 2D}(z, r_j))}{\mu(B(z, r_j))} = 1 - \frac{\mu(A_\mu^{\eta/\kappa, 2D}(z, r_j))}{\mu(B(z, r_j))}.$$

On the other hand, it follows from Observation 9.11 that

$$\mu(B(z, s_j)) \leq e^\kappa e^{-\kappa l_{n_j}} \quad \text{and} \quad \mu(B(z, r_j)) \geq e^{-\kappa l_{k_j}}.$$

This yields

$$\frac{\mu(B(z, s_j))}{\mu(B(z, r_j))} \leq e^\kappa \exp(-\kappa(l_{n_j} - l_{k_j})) \leq e^\kappa \exp(-\kappa(n_j - k_j)) \leq e^\kappa e^{-2\kappa} = e^{-\kappa}.$$

Along with (10.2) this implies that

$$(10.3) \quad \frac{\mu(A_\mu^{\eta/\kappa, 2D}(z, r_j))}{\mu(B(z, r_j))} \geq 1 - e^{-\kappa}.$$

However, since  $\lim_{j \rightarrow \infty} r_j = 0$ , since the measure  $\mu$  is (WBT), and since  $\kappa > 0$  was taken so small that  $\eta/\kappa > \beta_\mu(z)$ , we conclude that (10.3) may hold for finitely many integers  $j \geq 0$  only, and the proof of Lemma 10.5 is complete.  $\square$

As an immediate consequence of this lemma and Observation 10.3, we get the following.



**Lemma 10.6.** *For all integers  $k \geq 0$  large enough, if  $n - k \geq 2$ , then*

$$W^-(z, s) \subseteq W^-(z, r) \quad \text{and} \quad W^+(z, s) \subseteq W^+(z, r)$$

*for all  $r \in \mathcal{R}_k$  and all  $s \in \mathcal{R}_n$ .*

Now we shall prove the following.

**Proposition 10.7.** *Let  $\mathcal{S}$  be a conformal GDMS. Let  $\mu$  be a Borel probability measure supported on  $\overline{J}_{\mathcal{S}}$ . Suppose that  $\mu$  is (WBT) at some point  $z \in J_{\mathcal{S}}$  which is not an atom of  $\mu$ . Let  $\mathcal{R}$  be an arbitrary countable set of positive reals containing 0 in its closure. Then there exists  $(n_j)_{j=0}^{\infty}$ , a strictly increasing sequence of non-negative integers with the following properties.*

- (a)  $n_{j+1} - n_j \leq 4$ ,
- (b)  $n_{j+1} - n_j \geq 2$ ,
- (c) *The set  $\mathcal{R} \cap \mathcal{R}_{n_j} \neq \emptyset$  for infinitely many  $j$ s.*

*Proof.* We construct the sequence  $(n_j)_{j=0}^{\infty}$  inductively. Assume without loss of generality that  $r_0 = 2\text{diam}(\overline{J}_{\mathcal{S}})$  and set  $n_0 := 0$ . For the inductive step suppose that  $n_j \geq 0$  with some  $j \geq 0$  has been constructed. Look at the set  $\mathcal{R}(n_j + 2, n_j + 4)$ . If

$$\{l_k : n_j + 2 \leq k \leq n_j + 4\} \cap \{N_{\kappa}(z, r) : r \in \mathcal{R}\} \neq \emptyset,$$

take  $n_{j+1}$  to be an arbitrary number from  $\{n_j + 2, n_j + 3, n_j + 4\}$  such that

$$l_{n_{j+1}} \in \{N_{\kappa}(z, r) : r \in \mathcal{R}\}.$$

If, on the other hand,

$$\{l_k : n_j + 2 \leq k \leq n_j + 4\} \cap \{N_{\kappa}(z, r) : r \in \mathcal{R}\} = \emptyset,$$

set

$$n_{j+1} = n_j + 2.$$

Properties (a) and (b) are immediate from our construction. In order to prove (c) suppose on the contrary that

$$\mathcal{R} \cap \bigcup_{j=p}^{\infty} \mathcal{R}_{n_j} = \emptyset$$

with some  $p \geq 0$ . This yields  $n_{j+1} = n_j + 2$  for all  $j \geq p$ , i.e.  $n_j = n_p + 2(j - p)$  and

$$\bigcup_{j=p}^{\infty} \{l_k : n_p + 2(j + 1 - p) \leq k \leq n_p + 2(j + 2 - p)\} \cap \{N_{\kappa}(z, r) : r \in \mathcal{R}\} = \emptyset$$

But

$$\bigcup_{j=p}^{\infty} \{l_k : n_p + 2(j + 1 - p) \leq k \leq n_p + 2(j + 2 - p)\} = \{l_k : k \geq n_p + 2\}.$$

Thus,  $N_\kappa(z, r) \leq n_p + 1$  for all  $r \in \mathcal{R}$ . By Observation 9.11 this gives that  $\mu(B(z, r)) \geq \exp(-\kappa(n_p + 1))$  for all  $r \in \mathcal{R}$ , contrary to the facts that  $0 \in \overline{\mathcal{R}}$  and that  $z$  is not an atom of  $\mu$ . We are done.  $\square$

Now, for every  $j \geq 0$  fix arbitrarily  $r_j \in \mathcal{R}_{n_j}$  requiring in addition that  $r_j \in \mathcal{R}$  if  $\mathcal{R} \cap \mathcal{R}_{n_j} \neq \emptyset$ . Set

$$(10.4) \quad U_{l_{n_j}}^-(z) := \pi^{-1}(W^-(z, r_j)) \quad \text{and} \quad U_{l_{n_j}}^+(z) := \pi^{-1}(W^+(z, r_j)).$$

These sets are well defined as the function  $l : \mathbb{N} \rightarrow \mathbb{N}$  is 1-to-1 and, by (b), the function  $j \mapsto n_j$  is also 1-to-1. Furthermore, for every  $j \geq 0$  and every  $l_{n_j} \leq k < l_{n_{j+1}}$ , define

$$(10.5) \quad U_k^\pm(z) := U_{l_{n_j}}^\pm(z).$$

In this way we have well-defined two sequences of open neighborhoods of  $\pi^{-1}(z)$ . We shall prove the following.

**Proposition 10.8.** *With hypotheses exactly as in Proposition 10.7, both  $(U_k^\pm(z))_{k=0}^\infty$  are descending sequences of open subsets of  $E_A^\infty$  satisfying conditions (U0)–(U2).*

*Proof.* (U0) is immediate from the very definition. If  $k \geq 0$  and then  $j = j_k \geq 0$  is uniquely chosen so that  $l_{n_j} \leq k < l_{n_{j+1}}$ , then  $U_k^\pm(z) := U_{l_{n_j}}^\pm(z)$ , and both sets are disjoint unions of cylinders of length  $n_j$  by Observation 10.4 and since  $r_j \in \mathcal{R}_{n_j}$ , so also of length  $k$  as  $k \geq l_{n_j}$ . Thus (U1) holds. That both sequences  $(U_k^\pm(z))_{k=0}^\infty$  are descending follows immediately from Lemmas 10.6, property (b) of Proposition 10.7, and formulas (10.5) and (10.4). Applying formulas (10.5) and (10.4) along with Proposition 10.7 (b), Lemma 10.5, Observation 10.3, Observation 9.11, Lemma 10.2, and Proposition 10.7 (a), we get

$$(10.6) \quad \begin{aligned} \mu(U_k^\pm(z)) &\leq \mu(U_k^+(z)) = \mu(U_{l_{n_j}}(z)) = \mu(\pi^{-1}(W^+(z, r_j))) \leq \mu(\pi^{-1}(W^+(z, r_{j-1}))) \\ &= \mu(\pi^{-1}(W^-(z, r_{j-1}))) \leq \mu(\pi^{-1}(B(z, r_{j-1}))) \leq e^\kappa \exp(-\kappa N_\kappa(z, r_{j-1})) \\ &= e^\kappa e^{-l_{n_{j-1}}} = e^\kappa e^{-l_{n_{j+1}}} \exp(\kappa(l_{n_{j+1}} - l_{n_{j-1}})) \\ &\leq e^\kappa e^{-l_{n_{j+1}}} \exp(\kappa \Delta l(z)(n_{j+1} - n_{j-1})) \leq e^\kappa e^{8\kappa \Delta l(z)} \exp(-\kappa l_{n_{j+1}}) \\ &\leq \exp(\kappa((1 + 8\Delta l(z)))e^{-\kappa k}, \end{aligned}$$

and thus condition (U2) is satisfied with any  $\rho \in (e^{-\kappa}, 1)$  sufficiently close to 1. The proof is complete.  $\square$

**Proposition 10.9.** *With hypotheses exactly as in Proposition 10.7, both  $(U_k^\pm(z))_{k=0}^\infty$  satisfy condition (U3). If in addition either  $z$  is not pseudo-periodic for  $\mathcal{S}$  or it is uniquely periodic and  $z \in \text{Int}X$ , then (U5) holds. In the former case also (U4) holds while in the latter case it holds if in addition  $\mu$  is an equilibrium state of the amalgamated function of a summable Hölder continuous system of functions.*

*Proof.* With the same arguments as in (10.6) we get that

$$(10.7) \quad \pi^{-1}(z) \subseteq \bigcap_{k=0}^{\infty} \overline{U_n^-(z)} \subseteq \bigcap_{k=0}^{\infty} \overline{U_n^+(z)} \subseteq \bigcap_{j=1}^{\infty} \pi^{-1}(\overline{B(z, r_{j-1})}) \subseteq \pi^{-1}(z).$$

So (U3) holds as  $\pi^{-1}(z)$  is a finite set. Assume now that  $z$  is not pseudo-periodic. Then condition (U4A) holds because of Lemma 8.1 and (10.7), while (U5) directly follows from Lemma 8.2 and the inclusion  $U_{l_{n_j}}^{\pm}(z) \subseteq \pi^{-1}(B(z, r_{j-1}))$ .

Assume in turn that  $z \in \text{Int}X$  is uniquely periodic point of  $\mathcal{S}$  with prime period  $p$ . Then  $U_{\infty}$  consists of a periodic point, call it  $\xi$ , of period  $p$  because of (10.7). So,  $\xi = \tau^{\infty}$  for a unique point  $\tau \in E_A^{\infty}$ . Condition (U5) directly follows from Lemma 8.3. Now we shall show that the sequence  $(U_i^+(z))_{i=0}^{\infty}$  satisfies the property (U4B). Indeed, without loss of generality we may assume that  $i = l_k$ , where  $k = n_j$ ,  $j \geq 0$ . Take an arbitrary  $\omega \in U_{l_k}^+(z)$ . This means that  $\omega|_{l_k} \in E_A^{l_k}$  and  $\varphi_{\omega|_{l_k}}(J) \cap B(z, r_j) \neq \emptyset$ . Then

$$\varphi_{\tau\omega|_{l_k}}(J) \cap B(z, r_j) \supseteq \varphi_{\tau\omega|_{l_k}}(J) \cap \varphi_{\tau}(B(z, r_j)) = \varphi_{\tau}(\varphi_{\omega|_{l_k}}(J) \cap B(z, r_j)) \neq \emptyset.$$

Hence,  $\varphi_{\omega|_{l_k}}(J) \cap B(z, r_j) \neq \emptyset$ , meaning that  $\tau\omega \in U_{l_k}^+(z)$ . So, the inclusion  $\tau U_{l_k}^+(z) \subseteq U_{l_k}^+(z)$  has been proved and (6.1) of (U4B) holds for the sequence  $(U_i^+(z))_{i=0}^{\infty}$ .

In order to establish (6.1) of (U4B) for the sequence  $(U_i^-(z))_{i=0}^{\infty}$ , recall that  $\eta > 0$  is so small that  $\|\varphi'_{\omega}\| \leq e^{-\eta|\omega|}$  for all  $\omega \in E_A^*$ . Take now  $\kappa > 0$  as small as previously required and furthermore so small that  $\beta\eta\kappa^{-1} > 2$ . On the other hand, for every  $k := n_j$ ,  $j \geq 1$ , we have

$$(10.8) \quad \varphi_{\tau}(W^-(z, r_j)) \subseteq \varphi_{\tau}(B(z, r_j)) \subseteq B(z, |\varphi'_{\tau}(z)|r_j) \subseteq B(z, e^{-\eta|\tau|}r_j).$$

On the other hand, by (10.1) and the definition of  $l_{n_j}$ , we have that

$$\begin{aligned} W^-(z, r_j) &\supseteq B(z, r_j - De^{-\eta l_j}) \supseteq B(z, r_j - D\mu^{\eta/\kappa}(B(z, r_j))) \\ &\supseteq B(z, r_j - DC^{\eta/\kappa}r_j^{\beta\eta/\kappa}) \\ &\supseteq B(z, e^{-\eta|\tau|}r_j) \end{aligned}$$

provided that  $\kappa > 0$  is taken sufficiently small (independently of  $j$ ). Along with (10.8) this gives,

$$\varphi_{\tau}(W^-(z, r_j)) \subseteq W^-(z, r_j).$$

Hence,

$$\begin{aligned} \pi(\tau U_{l_k}^-(z)) &= \pi(\tau \pi^{-1}(W^-(z, r_j))) = \varphi_{\tau}(\pi(\pi^{-1}(W^-(z, r_j)))) \\ &= \varphi_{\tau}(W^-(z, r_j)). \end{aligned}$$

Thus

$$\tau U_{l_k}^-(z) \subseteq \pi^{-1}(W^-(z, r_j)) = U_{l_k}^-(z).$$

Thus, the part (6.1) of (U4B) is established. In order to prove (6.2) of (U4B), let  $k \geq 0$  and  $j_k \geq 0$  be as in the proof of Proposition 10.8. The proof of this proposition gives that

$$U_k^\pm(z) \subseteq \pi^{-1}(W^-(z, r_{j_k-1})).$$

Since we now assume that  $\varphi(\omega) = f^{\omega_0}(\pi(\sigma(\omega)))$ ,  $\omega \in E_A^\infty$ , where  $(f^e)_{e \in E}$  is a Hölder continuous summable system of functions, condition (6.2) of (U4B) follows from continuity of the function  $f^{\tau_0} : X_{t(\tau_0)} \rightarrow \mathbb{R}$  and the fact that  $\lim_{k \rightarrow \infty} j_k = +\infty$ . The proof of our proposition is complete.  $\square$

Now, we are in position to prove the following main result of this section, which is also one of the main results of the entire paper. Recall that the lower and upper escape rates  $\underline{R}_\mu$  and  $\overline{R}_\mu$  have been defined by formulas (6.13) and (6.14).

**Theorem 10.10.** *Let  $\mathcal{S} = \{\varphi_e\}_{e \in E}$  be a finitely primitive Conformal Graph Directed Markov System. Let  $\varphi : E_A^\infty \rightarrow \mathbb{R}$  be a Hölder continuous summable potential. As usual, denote its equilibrium/Gibbs state by  $\mu_\varphi$ . Assume that the measure  $\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}$  is (WBT) at a point  $z \in J_{\mathcal{S}}$ . If  $z$  is either*

(a) *not pseudo-periodic,*

*or*

(b) *uniquely periodic, it belongs to  $\text{Int}X$  (and  $z = \pi(\xi^\infty)$  for a (unique) irreducible word  $\xi \in E_A^*$ ), and  $\varphi$  is the amalgamated function of a summable Hölder continuous system of functions,*

*then, with  $\underline{R}_{\mathcal{S},\varphi}(B(z, \varepsilon)) := \underline{R}_{\mu_\varphi}(\pi_{\mathcal{S}}^{-1}(B(z, \varepsilon)))$  and  $\overline{R}_{\mathcal{S},\varphi}(B(z, \varepsilon)) := \overline{R}_{\mu_\varphi}(\pi_{\mathcal{S}}^{-1}(B(z, \varepsilon)))$ , we have that*

$$(10.9) \quad \lim_{\varepsilon \rightarrow 0} \frac{\underline{R}_{\mathcal{S},\varphi}(B(z, \varepsilon))}{\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}(B(z, \varepsilon))} = \lim_{\varepsilon \rightarrow 0} \frac{\overline{R}_{\mathcal{S},\varphi}(B(z, \varepsilon))}{\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}(B(z, \varepsilon))} =$$

$$= d_\varphi(z) := \begin{cases} 1 & \text{if (a) holds} \\ 1 - \exp(S_p \varphi(\xi) - pP(\varphi)) & \text{if (b) holds,} \end{cases}$$

*where in (b),  $\{\xi\} = \pi_{\mathcal{S}}^{-1}(z)$  and  $p \geq 1$  is the prime period of  $\xi$  under the shift map.*

*Proof.* Assume for a contradiction that (10.9) does not hold. This means that there exists a strictly decreasing sequence  $s_n(z) \rightarrow 0$  of positive reals such that at least one of the sequences

$$\left( \frac{\underline{R}_{\mathcal{S},\varphi}(B(z, s_n(z)))}{\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}(B(z, s_n(z)))} \right)_{n=0}^\infty \quad \text{or} \quad \left( \frac{\overline{R}_{\mathcal{S},\varphi}(B(z, s_n(z)))}{\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}(B(z, s_n(z)))} \right)_{n=0}^\infty$$

does not have  $d_\varphi(z)$  as its accumulation point. Let

$$\mathcal{R} := \{s_n(z) : n \geq 0\}.$$

Let  $(U_n^\pm(z))_{n=0}^\infty$  be the corresponding sequence of open subsets of  $E_A^\infty$  produced in formula (10.5). Then, because of both Proposition 10.8 and Proposition 10.9, Proposition 6.7

applies to give

$$(10.10) \quad \lim_{n \rightarrow \infty} \frac{R_{\mu_\varphi}(U_n^\pm(z))}{\mu_\varphi(U_n^\pm(z))} = d_\varphi(z).$$

Let  $(n_j)_{j=0}^\infty$  be the sequence produced in Proposition 10.7 with the help of  $\mathcal{R}$  defined above. By this proposition there exists an increasing sequence  $(j_k)_{k=0}^\infty$  such that  $\mathcal{R} \cap \mathcal{R}_{n_{j_k}} \neq \emptyset$  for all  $k \geq 1$ . For every  $k \geq 1$  pick one element  $r_k \in \mathcal{R} \cap \mathcal{R}_{n_{j_k}}$ . Set  $q_k := l_{n_{j_k}}$ . By Observation 10.3 and formula (10.4), we then have

$$(10.11) \quad \begin{aligned} \frac{R_{\mu_\varphi}(U_{q_k}^-(z))}{\mu_\varphi(U_{q_k}^-(z))} \cdot \frac{\mu_\varphi(U_{q_k}^-(z))}{\mu_\varphi(B(z, r_k))} &\leq \frac{\underline{R}_{\mathcal{S}, \varphi}(B(z, r_k))}{\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}(B(z, r_k))} \leq \frac{\overline{R}_{\mathcal{S}, \varphi}(B(z, r_k))}{\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}(B(z, r_k))} \leq \\ &\leq \frac{R_{\mu_\varphi}(U_{q_k}^+(z))}{\mu_\varphi(U_{q_k}^+(z))} \cdot \frac{\mu_\varphi(U_{q_k}^+(z))}{\mu_\varphi(B(z, r_k))}. \end{aligned}$$

But, since  $\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}$  is (WBT) at  $z$ , it is (DBT) at  $z$  by Proposition 9.12, and it therefore follows from (9.4) along with formulas (10.1) and (10.4) that

$$\lim_{k \rightarrow \infty} \frac{\mu_\varphi(U_{q_k}^-(z))}{\mu_\varphi(B(z, r_k))} = 1 = \lim_{k \rightarrow \infty} \frac{\mu_\varphi(U_{q_k}^+(z))}{\mu_\varphi(B(z, r_k))}.$$

Inserting this to (10.10) and (10.11), yields:

$$\lim_{k \rightarrow \infty} \frac{\underline{R}_{\mathcal{S}, \varphi}(B(z, r_k))}{\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}(B(z, r_k))} = \lim_{k \rightarrow \infty} \frac{\overline{R}_{\mathcal{S}, \varphi}(B(z, r_k))}{\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}(B(z, r_k))} = d_\varphi(z).$$

Since  $r_k \in \mathcal{R}$  for all  $k \geq 1$ , this implies that  $d_\varphi(z)$  is an accumulation point of both sequences  $(\underline{R}_{\mathcal{S}, \varphi}(B(z, r_k))/\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}(B(z, r_k)))_{n=1}^\infty$ ,  $(\overline{R}_{\mathcal{S}, \varphi}(B(z, r_k))/\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}(B(z, r_k)))_{n=1}^\infty$ , and this contradiction finishes the proof of Theorem 10.10.  $\square$

Now, as an immediate consequence of Theorem 10.10 and Theorem 9.7, we get the following.

**Theorem 10.11.** *Assume that  $\mathcal{S}$  is a finitely primitive conformal GDMS whose limit set  $J_{\mathcal{S}}$  is geometrically irreducible. Let  $\varphi : E_A^\infty \rightarrow \mathbb{R}$  be a Hölder continuous strongly summable potential. As usual, denote its equilibrium/Gibbs state by  $\mu_\varphi$ . Then*

$$(10.12) \quad \lim_{\varepsilon \rightarrow 0} \frac{\underline{R}_{\mathcal{S}, \varphi}(B(z, \varepsilon))}{\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}(B(z, \varepsilon))} = \lim_{\varepsilon \rightarrow 0} \frac{\overline{R}_{\mathcal{S}, \varphi}(B(z, \varepsilon))}{\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}(B(z, \varepsilon))} = 1$$

for  $\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}$ -a.e. point  $z$  of  $J_{\mathcal{S}}$ .

In the realm of finite alphabets  $E$ , by virtue of Theorem 10.10 and both Theorem 9.9 and Theorem 9.10, we get the following stronger result.

**Theorem 10.12.** *Let  $\mathcal{S} = \{\varphi_e\}_{e \in E}$  be a primitive Conformal Graph Directed Markov System with a finite alphabet  $E$  acting in the space  $\mathbb{R}^d$ ,  $d \geq 1$ . Assume that either  $d = 1$  or that the system  $\mathcal{S}$  is geometrically irreducible. Let  $\varphi : E_A^\infty \rightarrow \mathbb{R}$  be a Hölder continuous potential. As usual, denote its equilibrium/Gibbs state by  $\mu_\varphi$ . Let  $z \in J_{\mathcal{S}}$  be arbitrary. If either  $z$  is*

- (a) *not pseudo-periodic,*  
*or*  
 (b) *uniquely periodic, it belongs to  $\text{Int}X$  (and  $z = \pi(\xi^\infty)$  for a (unique) irreducible word  $\xi \in E_A^*$ ), and  $\varphi$  is the amalgamated function of a summable Hölder continuous system of functions,*

then

$$(10.13) \quad \lim_{\varepsilon \rightarrow 0} \frac{R_{\mathcal{S}, \varphi}(B(z, \varepsilon))}{\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}(B(z, \varepsilon))} = \lim_{\varepsilon \rightarrow 0} \frac{\overline{R}_{\mathcal{S}, \varphi}(B(z, \varepsilon))}{\mu_\varphi \circ \pi_{\mathcal{S}}^{-1}(B(z, \varepsilon))} =$$

$$= d_\varphi(z) := \begin{cases} 1 & \text{if (a) holds} \\ 1 - \exp(S_p \varphi(\xi) - pP(\varphi)) & \text{if (b) holds,} \end{cases}$$

where in (b),  $\{\xi\} = \pi_{\mathcal{S}}^{-1}(z)$  and  $p \geq 1$  is the prime period of  $\xi$  under the shift map.

## 11. THE DERIVATIVES $\lambda'_n(t)$ AND $\lambda''_n(t)$ OF LEADING EIGENVALUES

In this section we have  $\mathcal{S} = \{\varphi_e\}_{e \in E}$ , a finitely primitive strongly regular conformal graph directed Markov system. We keep a parameter  $t > \theta_{\mathcal{S}}$  and consider the Hölder continuous summable potential  $\varphi_t : E_A^\infty \rightarrow \mathbb{R}$  given by the formula

$$\varphi_t(\omega) := t \log |\varphi'_{\omega_0}(\pi(\sigma(\omega)))|.$$

We further assume that a sequence  $(U_n)_{n=0}^\infty$  of open subsets of  $E_A^\infty$  is given satisfying the conditions (U0)-(U5). The eigenvalues  $\lambda$  and  $\lambda_n$  along with other objects associated to the potential  $\varphi_t$  are now indicated with the letter/number  $t$ .

Our goal in this section is to calculate the asymptotic behavior of derivatives  $\lambda'_n(t)$  and  $\lambda''_n(t)$  of leading eigenvalues of perturbed operators  $\mathcal{L}_{t,n}$  when the integer  $n \geq 0$  diverges to infinity and the parameter  $t$  approaches  $b_{\mathcal{S}}$ . This is a particularly tedious and technically involved task, partially due to unboundedness of the function  $\varphi_t$  in the supremum norm and partially due to lack of uniform topological mixing on the sets  $K_z(\varepsilon)$  introduced below.

The main theorems of this section form the crucial ingredients in the escape rates considerations of the next section, i.e. Section 12.

We start with the following.

**Theorem 11.1.** *For every  $0 \leq n \leq +\infty$ , the function  $(\theta_{\mathcal{S}}, +\infty) \ni t \mapsto \lambda_n(t) \in (0, +\infty)$  is real analytic and*

$$(11.1) \quad \lambda'(t) = \lim_{n \rightarrow \infty} \lambda'_n(t).$$

*Proof.* By extending the transfer operators  $\mathcal{L}_{t,n} : \mathcal{B}_\theta \rightarrow \mathcal{B}_\theta$  in the natural way to complex operators for all  $t \in \mathbb{C}$  with  $\text{Re}(t) < \theta_{\mathcal{S}}$ , and applying Kato-Rellich Perturbation Theorem (see [36]), along with Proposition 5.2, we see that for every  $0 \leq n \leq +\infty$  there exists  $V_n$ , an open neighborhood of  $(\theta_{\mathcal{S}}, +\infty)$ , such that each function  $(\theta_{\mathcal{S}}, +\infty) \ni t \mapsto \lambda_n(t) \in (0, +\infty)$  extends (and we keep the same symbol  $\lambda_n$  for this extension) to a holomorphic function

from  $V_n$  to  $\mathbb{C}$ , and also each function  $(\theta_S, +\infty) \ni t \mapsto Q_n^{(t)} \mathbb{1} \in \mathcal{B}_\theta$  extends to a holomorphic function from  $V_n$  to  $\mathbb{C}$  belonging to  $\mathcal{B}_\theta$ . Denote these latter extensions by

$$g_n : V_n \rightarrow \mathbb{C}, \quad n \geq 0.$$

It is also a part of Kato-Rellich Theorem that

$$(11.2) \quad \mathcal{L}_{t,n} g_n(t) = \lambda_n(t) g_n(t)$$

for all  $0 \leq n \leq +\infty$  and all  $t \in V_n$ . In particular, all the functions  $(\theta_S, +\infty) \ni t \mapsto \lambda_n(t) \in (0, +\infty)$ ,  $0 \leq n \leq +\infty$ , are real analytic. In order to prove (11.1), we shall derive first a "thermodynamical" formula for  $\lambda'_n(t)$ . Differentiating both sides of (11.2), we obtain

$$(11.3) \quad \mathcal{L}'_{t,n} g_n(t) + \mathcal{L}_{t,n} g'_n(t) = \lambda'_n(t) g_n(t) + \lambda_n(t) g'_n(t),$$

where

$$(11.4) \quad \mathcal{L}'_{t,n}(u)(\omega) := \sum_{E: A_{E\omega_0}=1} \mathbb{1}_{U_n^c}(e\omega) u(e\omega) \log |\varphi'_e(\pi(u))| \cdot |\varphi'_e(\pi(u))|^t,$$

and all four terms involved in (11.3) belong to  $\mathcal{B}_\theta$ . Applying the operator  $Q_n^{(t)}$  to both sides of this equation, we get

$$Q_n^{(t)}(\mathcal{L}'_{t,n} g_n(t)) + Q_n^{(t)} \mathcal{L}_{t,n}(g'_n(t)) = \lambda'_n(t) Q_n^{(t)}(g_n(t)) + \lambda_n(t) Q_n^{(t)}(g'_n(t)).$$

Since

$$Q_n^{(t)}(g_n(t)) = g_n(t)$$

and

$$Q_n^{(t)} \mathcal{L}_{t,n}(g'_n(t)) = \mathcal{L}_{t,n} Q_n^{(t)}(g'_n(t)) = \lambda_n(t) Q_n^{(t)}(g'_n(t)),$$

we thus get

$$(11.5) \quad \lambda'_n(t) g_n(t) = Q_n^{(t)}(\mathcal{L}'_{t,n} g_n(t)).$$

Since in addition  $Q_n^{(t)}$  is a projector onto the 1-dimensional space  $\mathbb{C}g_n(t)$ , this operator gives rise to a unique bounded linear functional

$$\nu_{t,n} : \mathcal{B}_\theta \rightarrow \mathbb{C}$$

determined by the property that

$$Q_n^{(t)}(u) = \nu_{t,n}(u) g_n(t)$$

for every  $u \in \mathcal{B}_\theta$ . So, we can write (11.5) in the form

$$(11.6) \quad \lambda'_n(t) = \nu_{t,n}(\mathcal{L}'_{t,n} g_n(t)).$$

Now, writing

$$\ell(\omega) := \log |\varphi_{\omega_0}(\pi(\sigma(\omega)))|,$$

formula (11.4) readily gives

$$\mathcal{L}'_{t,n}(u) = \mathcal{L}_{t,n}(u\ell),$$

so that (11.5) takes on the form

$$(11.7) \quad \lambda'_n(t) = \nu_{t,n}(\mathcal{L}_{t,n}(\ell g_n(t))).$$

Keeping  $t \in (\theta_S, +\infty)$  for the rest of the proof set

$$\psi_n := \mathcal{L}_{t,n}(\ell g_n(t))$$

but remember that  $\psi_n$  depends on  $t$  too. Now, we have

$$\begin{aligned} Q_n^{(t)}(\psi_n) - Q^{(t)}(\psi) &= \nu_{t,n}(\psi_n)g_n(t) - \nu_t(\psi)g(t) \\ &= (\nu_{t,n}(\psi_n) - \nu_t(\psi))g(t) + (g_n(t) - g(t))\nu_{t,n}(\psi_n). \end{aligned}$$

Hence, recalling that  $g(t) \equiv \mathbb{1}$ , we get

$$(\nu_{t,n}(\psi_n) - \nu_t(\psi))\mathbb{1} = Q_n^{(t)}(\psi_n) - Q^{(t)}(\psi) + (g(t) - g_n(t))\nu_{t,n}(\psi_n).$$

Therefore,

$$\begin{aligned} |\nu_{t,n}(\psi_n) - \nu_t(\psi)| &= \int |Q_n^{(t)}(\psi_n) - Q^{(t)}(\psi) + \nu_{t,n}(\psi_n)(g(t) - g_n(t))| d\nu_t \\ &\leq \int |Q_n^{(t)}(\psi_n) - Q^{(t)}(\psi)| d\nu_t + \nu_{t,n}(-\psi_n) \int |g_n(t) - g(t)| d\nu_t. \end{aligned}$$

But, because of Proposition 5.2 (h),

$$\begin{aligned} \int |g_n(t) - g(t)| d\nu_t &\leq \|g_n(t) - g(t)\|_* = \|Q_n^{(t)}(\mathbb{1}) - Q^{(t)}(\mathbb{1})\|_* \\ (11.8) \quad &\leq \left\| \left\| Q_n^{(t)}(\mathbb{1}) - Q^{(t)}(\mathbb{1}) \right\| \right\| \|\mathbb{1}\|_\theta \\ &= \left\| \left\| Q_n^{(t)}(\mathbb{1}) - Q^{(t)}(\mathbb{1}) \right\| \right\| \rightarrow 0 \end{aligned}$$

as  $n \rightarrow 0$ . Hence, in view (11.7), in order to conclude the theorem, it suffices to show that

$$(11.9) \quad \lim_{n \rightarrow \infty} \int |Q_n^{(t)}(\psi_n) - Q^{(t)}(\psi)| d\nu_t = 0$$

and

$$(11.10) \quad M := \sup_{n \geq 1} \{\nu_{t,n}(-\psi_n)\} < +\infty.$$

We first deal with the latter. It follows from Proposition 5.2 (f) that  $|\lambda_n(t)| \geq 1/2$  for all  $n \geq 1$  large enough. It therefore follows from Proposition 5.2 (c) and (e) along with Lemma 3.1 that

$$(11.11) \quad \|g_n(t)\|_\infty \leq \|g_n(t)\|_\theta = \|Q_n^{(t)}(\mathbb{1})\|_\theta \leq 2(\|\mathcal{L}_{t,n}\mathbb{1}\|_\theta + \|\Delta_{t,n}\mathbb{1}\|_\theta) \leq 2(1+C) < +\infty.$$

Now, since  $t > \theta_S$ , it directly follows from the inequality

$$(11.12) \quad |\log |\varphi'_e|| \leq \|\varphi'_e\|^{-\varepsilon}$$

for every  $e > 0$  and all  $e \in \mathbb{N}$  large enough that

$$\begin{aligned} \|\psi_n\|_\infty &= \|\mathcal{L}_{t,n}(\ell g_n(t))\|_\infty \leq \|g_n(t)\|_\infty \|\mathcal{L}_{t,n}\ell\|_\infty \\ (11.13) \quad &\leq 2(1+C)\|\mathcal{L}_{t,n}\ell\|_\infty \\ &\leq 2(1+C)\|\mathcal{L}_t\ell\|_\infty < +\infty \end{aligned}$$



for all  $n \geq 1$  (including infinity) large enough. In fact we will need a somewhat more general result, namely that for every  $\gamma \in \mathcal{B}_\theta$ ,

$$\begin{aligned}
 (11.14) \quad \|\mathcal{L}_{t,n}(\ell\gamma)\|_1 &\leq \|\mathcal{L}_{t,n}(\ell\gamma)\|_\infty \leq \|\mathcal{L}_{t,n}(\ell \cdot \|\gamma\|_\infty)\|_\infty \\
 &= \|\gamma\|_\infty \|\mathcal{L}_{t,n}(\ell)\|_\infty \\
 &\leq \|\mathcal{L}_{t,n}\ell\|_\infty \|\gamma\|_\theta \\
 &\leq \|\mathcal{L}_t\ell\|_\infty \|\gamma\|_\theta.
 \end{aligned}$$

Let us now estimate  $|\psi_n|_\theta$ . Write

$$(11.15) \quad \gamma_n := g_n(t) \mathbb{1}_{U_n^c}.$$

Then

$$(11.16) \quad \psi_n := \mathcal{L}_{t,n}(\ell g_n(t)) = \mathcal{L}_t(\ell \gamma_n).$$

We now will also proceed more generally than merely estimating  $|\psi_n|_\theta$ . We shall prove the following.

**Lemma 11.2.** *There exists a constant  $C > 0$  such that for every  $\gamma \in \mathcal{B}_\theta$ , we have that*

$$\|\mathcal{L}_t(\ell\gamma)\|_\theta \leq C \|\gamma\|_\theta.$$

*Proof.* By virtue of 11.14 it suffices to estimate  $|\mathcal{L}_t(\ell\gamma)|_\theta$ . Fix an integer  $m \geq 0$ ,  $\omega \in E_A^\infty$ , and  $\alpha, \beta \in [\omega]_m$ . Let  $e \in E$  be such that  $A_{e\alpha_0} = A_{e\beta_0} = 1$ . Then

$$\begin{aligned}
 &\left| \ell(e\beta)\gamma(e\beta)|\varphi'_e(\pi(\beta))|^t - \ell(e\alpha)\gamma(e\alpha)|\varphi'_e(\pi(\alpha))|^t \right| \\
 &= \left| \ell(e\beta)(\gamma(e\beta)|\varphi'_e(\pi(\beta))|^t - \gamma(e\alpha)|\varphi'_e(\pi(\alpha))|^t) + \gamma(e\alpha)|\varphi'_e(\pi(\alpha))|^t(\ell(e\beta) - \ell(e\alpha)) \right| \\
 &\leq \left| \ell(e\beta)(\gamma(e\beta)(|\varphi'_e(\pi(\beta))|^t - |\varphi'_e(\pi(\alpha))|^t) + |\varphi'_e(\pi(\alpha))|^t(\gamma(e\beta) - \gamma(e\alpha))) \right| \\
 &\quad + \text{osc}_{m+1}(\ell)(e\omega)\gamma(e\alpha)|\varphi'_e(\pi(\alpha))|^t \\
 &\leq A\theta^{2m}|\ell(e\beta)\gamma(e\beta)| \cdot |\varphi'_e(\pi(\beta))|^t + |\varphi'_e(\pi(\alpha))|^t \text{osc}_{m+1}(\gamma)(e\omega) + \\
 &\quad + A\theta^{2m}\gamma(e\alpha)|\varphi'_e(\pi(\alpha))|^t
 \end{aligned}$$

with some constant  $A \in (0, +\infty)$  and some constant  $\theta \in (0, 1)$  sufficiently close to 1. Hence, using also (11.14) and Lemma 3.1, we get

$$\begin{aligned}
 &|\mathcal{L}_t(\ell\gamma)(\beta) - \mathcal{L}_t(\ell\gamma)(\alpha)| \\
 &\leq A\theta^{2m}(|\mathcal{L}_t(\ell\gamma)(\beta) - \mathcal{L}_t(\ell\gamma)(\alpha)| + K^t \mathcal{L}_t(\text{osc}_{m+1}(\gamma))(\omega)) \\
 &\leq A\theta^{2m}(\|\mathcal{L}_t(\ell)\|_\infty \|\gamma\|_\theta + \|\mathcal{L}_t(\gamma)\|_\infty) + K^t \mathcal{L}_t(\text{osc}_{m+1}(\gamma))(\omega) \\
 &\leq A\theta^{2m}(\|\mathcal{L}_t(\ell)\|_\infty \|\gamma\|_\theta + \|\mathcal{L}_t(\gamma)\|_\theta) + K^t \mathcal{L}_t(\text{osc}_{m+1}(\gamma))(\omega) \\
 &\leq A\theta^{2m}(\|\mathcal{L}_t(\ell)\|_\infty \|\gamma\|_\theta + \|\mathcal{L}_t\|_\theta \|\gamma\|_\theta) + K^t \mathcal{L}_t(\text{osc}_{m+1}(\gamma))(\omega) \\
 &\leq A\theta^{2m}(C + 1 + \|\mathcal{L}_t(\ell)\|_\infty) \|\gamma\|_\theta + K^t \mathcal{L}_t(\text{osc}_{m+1}(\gamma))(\omega)
 \end{aligned}$$

where we know that  $\|\mathcal{L}_t\|_\theta \leq C + 1$ . Therefore,

$$\text{osc}(\mathcal{L}_t(\ell\gamma))(\omega) \leq A\theta^{2m}(1 + \|\mathcal{L}_t(\ell)\|_\infty)\|\gamma\|_\theta + K^t \mathcal{L}_t(\text{osc}_{m+1}(\gamma))(\omega).$$

Thus, after integrating against measure  $\nu_t$ , we get

$$\begin{aligned} \|\text{osc}_m(\mathcal{L}_t(\ell\gamma))\|_{L^1(\nu_t)} &\leq A\theta^{2m}(C + 1 + \|\mathcal{L}_t(\ell)\|_\infty)\|\gamma\|_\theta + K^t \int \mathcal{L}_t(\text{osc}_{m+1}(\gamma)) d\nu_t \\ &= A\theta^{2m}(C + 1 + \|\mathcal{L}_t(\ell)\|_\infty)\|\gamma\|_\theta + K^t \int \text{osc}_{m+1}(\gamma) d\nu_t \\ &\leq A\theta^{2m}(C + 1 + \|\mathcal{L}_t(\ell)\|_\infty)\|\gamma\|_\theta + K^t \theta^{-(m+1)}|\gamma|_\theta. \end{aligned}$$

Therefore,

$$\theta^{-2m}\|\text{osc}_m(\mathcal{L}_t(\ell\gamma))\|_{L^1(\nu_t)} \leq (A(C + 1 + \|\mathcal{L}_t(\ell)\|_\infty) + K^t \theta^{-1})\|\gamma\|_\theta.$$

Combining this with (11.14) we finally get

$$\|\mathcal{L}_t(\ell\gamma)\|_\theta \leq (\|\mathcal{L}_t\ell\|_\infty + A(C + 1 + \|\mathcal{L}_t(\ell)\|_\infty) + K^t \theta^{-1})\|\gamma\|_\theta.$$

So, the proof is complete.  $\square$

As a fairly straightforward consequence of this lemma we get the following.

**Corollary 11.3.** *There exists a constant  $C' > 0$  such that for every  $\gamma \in \mathcal{B}_\theta$  and all  $n \geq 1$ , we have that*

$$\|\mathcal{L}_{t,n}(\ell\gamma)\|_\theta \leq C'\|\gamma\|_\theta.$$

*Proof.* By virtue of Lemma 4.4 (with  $k = 1$ ) and Lemma 4.1 we get

$$\|\mathbb{1}_n \gamma\|_\theta \leq |\gamma|_\theta + (1 - \theta)^{-1}\|\gamma\|_* \leq (1 + 2(1 - \theta)^{-1})\|\gamma\|_\theta.$$

Of course,

$$\|\mathbb{1}_n \gamma\|_{L^1(\nu_t)} \leq \|\gamma\|_{L^1(\nu_t)} \leq \|\gamma\|_\theta.$$

Hence,

$$\|\mathbb{1}_n \gamma\|_\theta \leq 2(1 + (1 - \theta)^{-1})\|\gamma\|_\theta.$$

As

$$\mathcal{L}_{t,n}(\ell\gamma) = \mathcal{L}_t(\ell\gamma \mathbb{1}_n) = \mathcal{L}_t(\ell(\mathbb{1}_n \gamma)),$$

applying Lemma 11.2, we thus get

$$\|\mathcal{L}_{t,n}(\ell\gamma)\|_\theta = \|\mathcal{L}_t(\ell(\mathbb{1}_n \gamma))\|_\theta \leq 2C(1 + (1 - \theta)^{-1})\|\gamma\|_\theta.$$

The proof is complete.  $\square$

It immediately follows from this corollary, along with (11.16) and (11.11), that

$$(11.17) \quad \|\psi_n\|_\theta \leq M_1$$

with some constant  $M_1 \in (0, \infty)$  and all integers  $n \geq 0$ . But then by Proposition 5.2 (g),

$$\|Q_n^{(t)}(-\psi_n)\|_\theta \leq C\|\psi_n\|_\theta \leq CM_1.$$

Since, on the other hand,  $Q_n^{(t)}(-\psi_n) = \nu_{t,n}(-\psi_n)g_n(t)$ , and also, by (11.8),

$$\int g_n(t) d\nu_t \geq \frac{1}{2} \int g(t) d\nu_t = 1/2$$

for all  $n \geq 1$  sufficiently large, we thus conclude that

$$\begin{aligned} CM_1 &\geq \|Q_n^{(t)}(-\psi_n)\|_{L^1(\nu_t)} \geq \int |\nu_{t,n}(-\psi_n)g_n(t)| d\nu_t \\ &= |\nu_{t,n}(-\psi_n)| \int |g_n(t)| d\nu_t \\ &\geq \frac{1}{2} |\nu_{t,n}(-\psi_n)|. \end{aligned}$$

So,

$$|\nu_{t,n}(-\psi_n)| \leq 2CM_1,$$

and formula (11.10) is established.

Now we shall prove that (11.9) holds. Write, as usually,  $\|h\|_1 := \|h\|_{L^1(\nu_t)}$  for all  $h \in L^1(\nu_t)$ . With the use of (11.17) we then estimate

$$\begin{aligned} \|Q_n^{(t)}(\psi_n) - Q^{(t)}(\psi)\|_1 &= \|(Q_n^{(t)} - Q^{(t)})\psi_n + Q^{(t)}(\psi_n - \psi)\|_1 \\ &\leq \|(Q_n^{(t)} - Q^{(t)})\psi_n\|_1 + \|Q^{(t)}(\psi_n - \psi)\|_1 \\ &\leq \|(Q_n^{(t)} - Q^{(t)})\psi_n\|_* + \|Q^{(t)}\|_1 \|\psi_n - \psi\|_1 \\ &\leq \|Q_n^{(t)} - Q^{(t)}\| \|\psi_n\|_\theta + \|Q^{(t)}\|_1 \|\psi_n - \psi\|_1 \\ &\leq M_1 \|Q_n^{(t)} - Q^{(t)}\| + \|Q^{(t)}\|_1 \|\psi_n - \psi\|_1. \end{aligned}$$

Hence, applying Proposition 5.2 (h), we see that in order to prove that (11.9) holds, and by having done this, to conclude the proof of Theorem 11.1, it suffices to show that

$$(11.18) \quad \lim_{n \rightarrow \infty} \|\psi_n - \psi\|_1 = 0.$$

It is well-known, and follows easily from (11.12) that  $\ell \in L^p(\nu_t)$  for all real  $p > 0$ . Using Cauchy-Schwarz inequality we then estimate:

$$\begin{aligned} \|\psi_n - \psi\|_1 &= \|\mathcal{L}_t(\ell\gamma_n) - \mathcal{L}_t(\ell\gamma)\|_1 = \|\mathcal{L}_t(\ell\gamma_n - \ell\gamma)\|_1 = \|\ell(\gamma_n - \gamma)\|_1 \\ &\leq \|\ell\|_2 \|\gamma_n - \gamma\|_2 = \|\ell\|_2 \|g_n(t)\mathbb{1} - g(t)\|_2 \\ &= \|\ell\|_2 \|\mathbb{1}_n(\gamma_n(t) - \gamma(t)) + \gamma(t)(\mathbb{1}_n - \mathbb{1})\|_2 \\ &\leq \|\ell\|_2 (\|\mathbb{1}_n(\gamma_n(t) - \gamma(t))\|_2 + \|\gamma(t)\mathbb{1}_{U_n}\|_2) \\ &\leq \|\ell\|_2 (\|\gamma_n(t) - \gamma(t)\|_2 + \|\mathbb{1}_{U_n}\|_2) \\ &\leq \|\ell\|_2 (\|\gamma_n(t) - \gamma(t)\|_2 + \sqrt{\nu_t(U_n)}) \\ &\leq \|\ell\|_2 (\|\gamma_n(t) - \gamma(t)\|_4 + \sqrt{\nu_t(U_n)}) \end{aligned}$$

But  $\lim_{n \rightarrow \infty} \nu_t(U_n) = 0$  and  $\lim_{n \rightarrow \infty} \|g_n(t) - g(t)\|_4 = 0$  because of (11.8) and (11.11). Hence, the formula (11.18) holds and the proof of Theorem 11.1 is complete.  $\square$

Now our goal is to show that the derivatives  $\lambda_n''(t)$  are uniformly bounded above in appropriate domains of  $t$  and  $n$ . In order to do this we will need several auxiliary results. Our strategy is to apply the results of [11] for the family of operators

$$(\mathcal{L}_{t,n} : t \in (s - \delta, s + \delta), n \geq 0),$$

where  $s > \theta_S$  and  $\delta > 0$  is small enough. At the beginning the only normalization we assume is that  $\lambda_s = 1$  and  $g(s) \equiv \mathbb{1}$ . Later on for ease of expression we will also assume that  $\lambda_t = 1$  and  $g(s) \equiv \mathbb{1}$  for all  $t \in (s - \delta, s + \delta)$  and appropriate  $\delta > 0$  small enough. It is evident from the form of our potentials  $\varphi_t(\omega) := t \log |\varphi'_{\omega_0}(\pi(\sigma(\omega)))|$  that the distortion constants  $M_\varphi$  of Lemma 2.2 and Lemma 2.3 can be taken of common value for all  $t \in (0, 2s - \theta_S]$ . Denote this common constant by  $M_s$ . An inspection of the proof of Lemma 3.2 leads to the following.

**Lemma 11.4.** *For every  $\delta \in (0, s - \theta_S)$  there exists a constant  $C_\delta \in (0, +\infty)$  such that for every  $t \in [s - \delta, s + \delta]$ , every integer  $k \geq 0$ , and every  $g \in \mathcal{B}_\theta$ , we have*

$$|\mathcal{L}_t^k g|_\theta \leq C_\delta (\theta \lambda_t)^k |g|_\theta + \lambda_t^k \|g\|_1.$$

Since the function  $(\theta_S, +\infty) \ni t \mapsto \lambda_t$  is strictly decreasing, denoting  $\lambda_{s-\delta}$  by  $M$ , as an immediate consequence of Lemma 11.4 we get the following.

**Lemma 11.5.** *For every  $\delta \in (0, s - \theta_S)$ , every  $t \in [s - \delta, s + \delta]$ , every integer  $k \geq 0$ , and every  $g \in \mathcal{B}_\theta$ , we have*

$$|\mathcal{L}_t^k g|_\theta \leq C_\delta (\theta M)^k |g|_\theta + M^k \|g\|_1.$$

Lemma 4.2 directly translates into the following.

**Lemma 11.6.** *For every  $\delta \in (0, s - \theta_S)$ , every  $t \in [s - \delta, s + \delta]$ , every integer  $k \geq 0$ , and every  $n \geq 0$ , we have*

$$\|\mathcal{L}_{t,n}^k\|_* \leq \lambda_t^k \leq M^k.$$

The proof of Corollary 4.5 provides exact estimates of constants, and gives the following.

**Lemma 11.7.** *For every  $\delta \in (0, s - \theta_S)$ , every  $t \in [s - \delta, s + \delta]$ , every integer  $k \geq 0$ , every integer  $n \geq 0$ , and every  $g \in \mathcal{B}_\theta$ , we have*

$$\|\mathcal{L}_{t,n}^k g\|_\theta \leq (C_\delta + 1)(\theta M)^k \|g\|_\theta + (C_\delta + 1)(1 + \theta(1 - \theta)^{-1})M^k \|g\|_*.$$

From now on throughout the entire section we assume that condition (U2) holds in the following uniform version:

(U2\*) There exists  $\rho \in (0, 1)$  such that for some  $\delta > 0$  and for all integers  $n \geq 0$  we have

$$\sup \{ \mu_t(U_n) : t \in [s - \delta, s + \delta] \} \leq \rho^n.$$

We now have the following.

**Lemma 11.8.** *For every  $\delta \in (0, s - \theta_S)$ , every  $t \in [s - \delta, s + \delta]$  and every integer  $n \geq 0$ , we have*

$$|||\mathcal{L} - \mathcal{L}_n||| \leq 2\lambda_t(\rho^{1/q})^n \leq 2M\rho^{n/q}.$$

Now, Lemmas 11.6, 11.4, and 11.8, along with formula (5.8), and compactness (in fact finite dimensionality) of the operators  $\pi_k : \mathcal{B}_\theta \rightarrow \mathcal{B}_\theta$  imply that Theorem 1 in [11] along with all corollaries therein, applies to the family of operators

$$(\mathcal{L}_{t,n} : t \in (s - \delta, s + \delta), n \geq 0),$$

(i. e.  $\mathcal{L}_s$  corresponds to  $P_0$  and  $\mathcal{L}_{t,n}$  correspond to operators  $P_\varepsilon$ ) with

$$(t, n) \rightarrow s \Leftrightarrow t \rightarrow s \text{ and } n \rightarrow +\infty$$

to give the following extension of Proposition 5.2.

**Proposition 11.9.** *Fix  $s > \theta_S$ . Let the Perron-Frobenius operator  $\mathcal{L}_s : \mathcal{B}_\theta \rightarrow \mathcal{B}_\theta$  be normalized so that  $\lambda_s = 1$ . Then there exist  $\delta \in (0, s - \theta_S)$  sufficiently small and an integer  $n_s \geq 0$  sufficiently large such that for all  $(t, n) \in (s - \delta, s + \delta) \times \{n_s, n_s + 1, \dots\}$  there exist bounded operators  $Q_n^{(t)}, \Delta_n^{(t)} : \mathcal{B}_\theta \rightarrow \mathcal{B}_\theta$  and complex numbers  $\lambda_n(t) \neq 0$  with the following properties:*

- (a)  $\lambda_n(t)$  is a simple eigenvalue of the operator  $\mathcal{L}_{t,n} : \mathcal{B}_\theta \rightarrow \mathcal{B}_\theta$ .
- (b)  $Q_t^{(n)} : \mathcal{B}_\theta \rightarrow \mathcal{B}_\theta$  is a projector ( $Q_t^{(n)2} = Q_t^{(n)}$ ) onto the 1-dimensional eigenspace of  $\lambda_n(t)$ .
- (c)  $\mathcal{L}_{t,n} = \lambda_n(t)Q_t^{(n)} + \Delta_{t,n}$ .
- (d)  $Q_t^{(n)} \circ \Delta_{t,n} = \Delta_{t,n} \circ Q_t^{(n)} = 0$ .
- (e) There exist  $\kappa \in (0, 1)$  and  $C > 0$  (independent of  $(t, n) \in (s - \delta, s + \delta) \times \{n_s, n_s + 1, \dots\}$ ) such that

$$||\Delta_{t,n}^k||_\theta \leq C\kappa^k$$

for all  $k \geq 0$ . In particular,

$$||\Delta_{t,n}^k g||_\infty \leq ||\Delta_{t,n}^k g||_\theta \leq C\kappa^k ||g||_\theta$$

for all  $g \in \mathcal{B}_\theta$ .

- (f)  $\lim_{(t,n) \rightarrow s} \lambda_n(t) = 1$ .
- (g) Enlarging the above constant  $C > 0$  if necessary, we have

$$||Q_t^{(n)}||_\theta \leq C.$$

In particular,

$$||Q_t^{(n)} g||_\infty \leq ||Q_t^{(n)} g||_\theta \leq C ||g||_\theta$$

for all  $g \in \mathcal{B}_\theta$ .

- (h)  $\lim_{(t,n) \rightarrow s} |||Q_t^{(n)} - Q_s||| = 0$ .

Now we are ready to prove the following.

**Lemma 11.10.** *For every  $s > \theta_S$  there exists  $\eta \in (0, 1)$  such that*

$$M := \sup_{n \geq n_s} \sup \{ \lambda_n''(t) : t \in (s - \eta, s + \eta) \} < +\infty.$$

*Proof.* Throughout the whole proof we always assume that  $t \in (s - \delta, s + \delta)$  and  $n \geq n_s$ , where  $\delta > 0$  is the one produced in Proposition 11.9. Fix an integer  $N \geq 1$  and differentiate the eigenvalue equation

$$\mathcal{L}_{t,n}^N g_n(t) = \lambda_n^N(t) g_n(t)$$

with respect the variable  $t$  two times. This gives in turn

$$(\mathcal{L}_{t,n}^N)'(g_n(t)) + \mathcal{L}_{t,n}^N(g_n'(t)) = N \mathcal{L}_{t,n}^N \lambda_n'(t) \lambda_n^{N-1}(t) g_n(t) + \lambda_n^N(t) g_n'(t)$$

and

$$\begin{aligned} (\mathcal{L}_{t,n}^N)''(g_n(t)) + (\mathcal{L}_{t,n}^N)'(g_n'(t)) + (\mathcal{L}_{t,n}^N)'(g_n'(t)) + \mathcal{L}_{t,n}^N(g_n''(t)) &= \\ &= N(N-1) \lambda_n^{N-2}(t) (\lambda_n'(t))^2 g_n(t) + N \lambda_n^{N-1}(t) \lambda_n'(t) g_n'(t) + \\ &\quad + N \lambda_n^{N-1}(t) \lambda_n''(t) g_n(t) + N \lambda_n^{N-1}(t) \lambda_n'(t) g_n'(t) + \lambda_n^N(t) g_n''(t). \end{aligned}$$

Equivalently:

$$\begin{aligned} (\mathcal{L}_{t,n}^N)''(g_n(t)) + 2(\mathcal{L}_{t,n}^N)'(g_n'(t)) + \mathcal{L}_{t,n}^N(g_n''(t)) &= \\ &= N(N-1) \lambda_n^{N-2}(t) (\lambda_n'(t))^2 g_n(t) + 2N \lambda_n^{N-1}(t) \lambda_n'(t) g_n'(t) + \\ &\quad + N \lambda_n^{N-1}(t) \lambda_n''(t) g_n(t) + \lambda_n^N(t) g_n''(t). \end{aligned}$$

Noting that

$$Q_t^{(n)} \mathcal{L}_{t,n}^N(g_n''(t)) = \mathcal{L}_{t,n}^N Q_t^{(n)}(g_n''(t)) = \lambda_n^N(t) Q_t^{(n)}(g_n''(t))$$

and applying to both sides of this equality the linear operator  $Q_t^{(n)}$ , we get

$$\begin{aligned} Q_t^{(n)} (\mathcal{L}_{t,n}^N)''(g_n(t)) + 2Q_t^{(n)} (\mathcal{L}_{t,n}^N)'(g_n'(t)) &= \\ &= N(N-1) \lambda_n^{N-2}(t) (\lambda_n'(t))^2 g_n(t) + 2N \lambda_n^{N-1}(t) \lambda_n'(t) Q_t^{(n)}(g_n'(t)) + N \lambda_n^{N-1}(t) \lambda_n''(t) g_n(t). \end{aligned}$$

Now since  $(\mathcal{L}_{t,n}^N)'(g_n(t)) = \mathcal{L}_{t,n}^N(g_n(t) S_N \ell)$ , and so  $(\mathcal{L}_{t,n}^N)''(g_n(t)) = \mathcal{L}_{t,n}^N(g_n(t) (S_N \ell)^2)$ , and since also  $(\mathcal{L}_{t,n}^N)'(g_n'(t)) = \mathcal{L}_{t,n}^N(g_n'(t) S_N \ell)$ , we thus get

$$\begin{aligned} (11.19) \quad Q_t^{(n)} \mathcal{L}_{t,n}^N(g_n(t) (S_N \ell)^2) + 2Q_t^{(n)} \mathcal{L}_{t,n}^N(g_n'(t) S_N \ell) &= \\ &= N(N-1) \lambda_n^{N-2}(t) (\lambda_n'(t))^2 g_n(t) + 2N \lambda_n^{N-1}(t) \lambda_n'(t) Q_t^{(n)}(g_n'(t)) + N \lambda_n^{N-1}(t) \lambda_n''(t) g_n(t). \end{aligned}$$

We first deal with the term  $Q_t^{(n)} \mathcal{L}_{t,n}^N(g'_n(t) S_N \ell)$ . We have

$$\begin{aligned}
 (11.20) \quad Q_t^{(n)} \mathcal{L}_{t,n}^N(g'_n(t) S_N \ell) &= Q_t^{(n)} \mathcal{L}_{t,n}^N \left( g'_n(t) \sum_{j=0}^{N-1} \ell \circ \sigma^j \right) = \sum_{j=0}^{N-1} Q_t^{(n)} \mathcal{L}_{t,n}^N(g'_n(t) \ell \circ \sigma^j) \\
 &= \sum_{j=0}^{N-1} Q_t^{(n)} \mathcal{L}_{t,n}^{N-j}(\ell \mathcal{L}_{t,n}^j)'(g'_n(t)) \\
 &= \sum_{j=0}^{N-1} \mathcal{L}_{t,n}^{N-j-1} Q_t^{(n)} \mathcal{L}_{t,n}(\ell \mathcal{L}_{t,n}^j)'(g'_n(t)) \\
 &= \sum_{j=0}^{N-1} \lambda_n(t)^{N-j-1} Q_t^{(n)} \mathcal{L}_{t,n}(\ell \mathcal{L}_{t,n}^j)'(g'_n(t)).
 \end{aligned}$$

Now, by virtue of Proposition 11.9, particularly by its parts (c) and (e) and by Corollary 11.3, we get

$$\begin{aligned}
 &\left\| \mathcal{L}_{t,n}(\ell \mathcal{L}_{t,n}^j)'(g'_n(t)) - \lambda_n(t)^j \mathcal{L}_{t,n}(\ell Q_t^{(n)})'(g'_n(t)) \right\|_{\infty} \leq \\
 &\leq \left\| \mathcal{L}_{t,n}(\ell \mathcal{L}_{t,n}^j)'(g'_n(t)) - \lambda_n(t)^j \mathcal{L}_{t,n}(\ell Q_t^{(n)})'(g'_n(t)) \right\|_{\theta} \\
 &= \left\| \mathcal{L}_{t,n}(\ell(\mathcal{L}_{t,n}^j)'(g'_n(t)) - \lambda_n(t)^j Q_t^{(n)})'(g'_n(t)) \right\|_{\theta} \\
 &= \left\| \mathcal{L}_{t,n}(\ell \Delta_n^j)'(g'_n(t)) \right\|_{\theta} \leq C' \| \ell \|_{\theta} \leq C' C \kappa^j \| g'_n(t) \|_{\theta}.
 \end{aligned}$$

Therefore, by item (g) of Proposition 5.2 we get

$$(11.21) \quad \left\| Q_t^{(n)} \mathcal{L}_{t,n}(\ell \mathcal{L}_{t,n}^j)'(g'_n(t)) - \lambda_n^j(t) Q_t^{(n)} \mathcal{L}_{t,n}(\ell Q_t^{(n)})'(g'_n(t)) \right\|_{\infty} \leq C' C^2 \kappa^j \| g'_n(t) \|_{\theta}.$$

□

On the other hand, because of (11.7), we get

$$\begin{aligned}
 \lambda'_n(t) Q_t^{(n)}(g'_n(t)) &= \nu_{t,n}(g'_n(t)) \lambda'_n(t) g_n(t) = \nu_{t,n}(g'_n(t)) Q_t^{(n)} \mathcal{L}_{t,n}(\ell g_n(t)) \\
 &= Q_t^{(n)} \mathcal{L}_{t,n}(\ell \nu_{t,n}(g'_n(t)) g_n(t)) \\
 &= Q_t^{(n)} \mathcal{L}_{t,n}(\ell Q_t^{(n)}(g'_n(t)))
 \end{aligned}$$

Therefore, using (11.20) and (11.21), we get

$$\begin{aligned}
& \left\| \lambda_n^{1-N}(t) N^{-1} \left( Q_t^{(n)} \mathcal{L}_{t,n}(g'_n(t) S_N(\ell)) - N \lambda_n^{N-1}(t) \lambda'_n(t) Q_t^{(n)}(g'_n(t)) \right) \right\|_\infty = \\
& = \left\| \frac{1}{N} \sum_{j=0}^{N-1} \lambda_n(t)^{-j} Q_t^{(n)} \mathcal{L}_{t,n}(\ell \mathcal{L}_{t,n}^j(g'_n(t))) - Q_t^{(n)} \mathcal{L}_{t,n}(\ell Q_t^{(n)}(g'_n(t))) \right\|_\infty \\
& = \left\| \frac{1}{N} \sum_{j=0}^{N-1} \left[ \lambda_n(t)^{-j} Q_t^{(n)} \mathcal{L}_{t,n}(\ell \mathcal{L}_{t,n}^j(g'_n(t))) - Q_t^{(n)} \mathcal{L}_{t,n}(\ell Q_t^{(n)}(g'_n(t))) \right] \right\|_\infty \\
& \leq \frac{1}{N} \sum_{j=0}^{N-1} \left\| \lambda_n(t)^{-j} Q_t^{(n)} \mathcal{L}_{t,n}(\ell \mathcal{L}_{t,n}^j(g'_n(t))) - Q_t^{(n)} \mathcal{L}_{t,n}(\ell Q_t^{(n)}(g'_n(t))) \right\|_\infty \\
& \leq \frac{1}{N} \sum_{j=0}^{N-1} C' C^2 \|g'_n(t)\|_\theta (\kappa/\lambda_n(t))^j.
\end{aligned}$$

But, by Proposition 11.9, we have  $\kappa/\lambda_n(t) \leq \frac{2\kappa}{1+\kappa} < 1$  for all  $(t, n)$  sufficiently close to  $s$ . Therefore, for all such pairs  $(t, n)$ , we have

$$\lim_{N \rightarrow \infty} \left\| 2N^{-1} \lambda_n^{1-N}(t) \left( Q_t^{(n)} \mathcal{L}_{t,n}^N(g'_n(t) S_N(\ell)) - N \lambda_n^{N-1}(t) \lambda'_n(t) Q_t^{(n)}(g'_n(t)) \right) \right\|_\infty = 0$$

where the convergence, in the supremum norm  $\|\cdot\|_\infty$  is uniform with respect to all  $t$  sufficiently close to  $s$ . Inserting this to (11.19), we thus get  
(11.22)

$$\lambda_n''(t) g_n(t) = \lim_{N \rightarrow \infty} \left[ N^{-1} \lambda_n^{1-N}(t) Q_t^{(n)} \mathcal{L}_{t,n}^N(g_n(t) (S_N(\ell))^2) \right] - \lambda_n^{-1}(t) (N-1) g_n(t) (\lambda'_n(t))^2 g_n(t),$$

with the same meaning of convergence as above.



Let us first deal with the term  $Q_t^{(n)} \mathcal{L}_{t,n}^N(g_n(t)(S_N \ell)^2)$ . We have  
(11.23)

$$\begin{aligned}
& Q_t^{(n)} \mathcal{L}_{t,n}^N(g_n(t)(S_N \ell)^2) = \\
&= 2 \sum_{i=0}^{N-1} \sum_{j=i+1}^{N-1} Q_t^{(n)} \mathcal{L}_{t,n}^N(g_n(t) \ell \circ \sigma^i \cdot \ell \circ \sigma^j) + \sum_{j=0}^{N-1} Q_t^{(n)} \mathcal{L}_{t,n}^N(g_n(t) \ell^2 \circ \sigma^j) \\
&= \sum_{j=0}^{N-1} Q_t^{(n)} \mathcal{L}_{t,n}^N(g_n(t) \ell^2 \circ \sigma^j) + 2 \sum_{i=0}^{N-1} \sum_{j=i+1}^{N-1} Q_t^{(n)} \mathcal{L}_{t,n}^N(g_n(t) (\ell \cdot \ell \circ \sigma^{j-i}) \circ \sigma^i) \\
&= \sum_{j=0}^{N-1} Q_t^{(n)} \mathcal{L}_{t,n}^{N-j}(\ell^2 \mathcal{L}_{t,n}^j(g_n(t))) + 2 \sum_{i=0}^{N-1} \sum_{j=i+1}^{N-1} Q_t^{(n)} \mathcal{L}_{t,n}^{N-i}(\ell \cdot \ell \circ \sigma^{j-i} \mathcal{L}_{t,n}^i(g_n(t))) \\
&= \lambda_n^j(t) \sum_{j=0}^{N-1} Q_t^{(n)} \mathcal{L}_{t,n}^{N-j}(g_n(t) \ell^2) + 2 \lambda_n^i(t) \sum_{i=0}^{N-1} \sum_{j=i+1}^{N-1} Q_t^{(n)} \mathcal{L}_{t,n}^{N-i}(g_n(t) \ell \cdot \ell \circ \sigma^{j-i}) \\
&= \lambda_n^j(t) \sum_{j=0}^{N-1} Q_t^{(n)} \mathcal{L}_{t,n}^{N-j}(g_n(t) \ell^2) + 2 \lambda_n^i(t) \sum_{i=0}^{N-1} \sum_{j=i+1}^{N-1} Q_t^{(n)} \mathcal{L}_{t,n}^{N-j}(\ell \mathcal{L}_{t,n}^{j-i}(\ell g_n(t))) \\
&= \lambda_n^{N-1}(t) \sum_{j=0}^{N-1} Q_t^{(n)} \mathcal{L}_{t,n}(g_n(t) \ell^2) + 2 \sum_{i=0}^{N-1} \sum_{j=i+1}^{N-1} \lambda_n^{N+i-(j+1)}(t) Q_t^{(n)} \mathcal{L}_{t,n}(\ell \mathcal{L}_{t,n}^{j-i}(\ell g_n(t))) \\
&= \lambda_n^{N-1}(t) Q_t^{(n)} \mathcal{L}_{t,n}(g_n(t) \ell^2) + 2 \sum_{k=1}^{N-1} \lambda_n^{N-k-1}(t) (N-k) Q_t^{(n)} \mathcal{L}_{t,n}(\ell \mathcal{L}_{t,n}^k(\ell g_n(t))).
\end{aligned}$$

Now, using Proposition 11.9 (c) (and (d)) and denoting  $\psi_n = \mathcal{L}_{t,n} \ell g_n(t)$ , we get

$$\begin{aligned}
Q_t^{(n)} \mathcal{L}_{t,n}(\ell \mathcal{L}_{t,n}^k(\ell g_n(t))) &= Q_t^{(n)} \mathcal{L}_{t,n}(\ell \mathcal{L}_{t,n}^{k-1}(\mathcal{L}_{t,n} \ell g_n(t))) = Q_t^{(n)} \mathcal{L}_{t,n}(\ell \mathcal{L}_{t,n}^{k-1}(\psi_n)) \\
&= Q_t^{(n)} \mathcal{L}_{t,n}(\ell (\lambda_n^{k-1}(t) Q_t^{(n)}(\psi_n) + \Delta_n^{k-1}(\psi_n))) \\
&= \lambda_n^{k-1}(t) Q_t^{(n)} \mathcal{L}_{t,n}(\ell Q_t^{(n)}(\psi_n)) + Q_t^{(n)} \mathcal{L}_{t,n}(\Delta_n^{k-1}(\psi_n)) \\
&= \lambda_n^{k-1}(t) Q_t^{(n)} \mathcal{L}_{t,n}(\ell \nu_{t,n}(\psi_n) g_n(t)) + Q_t^{(n)} \mathcal{L}_{t,n}(\Delta_n^{k-1}(\psi_n)) \\
&= \lambda_n^{k-1}(t) \nu_{t,n}(\psi_n) Q_t^{(n)} \mathcal{L}_{t,n}(\ell g_n(t)) + Q_t^{(n)} \mathcal{L}_{t,n}(\Delta_n^{k-1}(\psi_n)) \\
&= \lambda_n^{k-1}(t) \nu_{t,n}(\psi_n) Q_t^{(n)}(\psi_n) + Q_t^{(n)} \mathcal{L}_{t,n}(\Delta_n^{k-1}(\psi_n)) \\
&= \lambda_n^{k-1}(t) (\nu_{t,n}(\psi_n))^2 g_n(t) + Q_t^{(n)} \mathcal{L}_{t,n}(\Delta_n^{k-1}(\psi_n)) \\
&= \lambda_n^{k-1}(t) (\lambda_n'(t))^2 g_n(t) + Q_t^{(n)} \mathcal{L}_{t,n}(\Delta_n^{k-1}(\psi_n)).
\end{aligned}$$

Therefore, using Proposition 11.9, we get

$$\begin{aligned}
2 \sum_{k=1}^{N-1} \lambda_n^{N-k-1}(t) (N-k) Q_t^{(n)} \mathcal{L}_{t,n}(\ell \mathcal{L}_{t,n}^k(\ell g_n(t))) &= \\
&= \lambda_n^{N-2}(t) N(N-1) (\lambda'_n(t))^2 g_n(t) + 2 \sum_{k=1}^{N-1} \lambda_n^{N-k-1}(t) Q_t^{(n)} \mathcal{L}_{t,n}(\Delta_n^{k-1}(\psi_n)) \\
&= \lambda_n^{N-2}(t) N(N-1) (\lambda'_n(t))^2 g_n(t) + 2 \sum_{k=1}^{N-1} \lambda_n^{N-k-1}(t) \mathcal{L}_{t,n} Q_t^{(n)}(\Delta_n^{k-1}(\psi_n)) \\
&= \lambda_n^{N-2}(t) N(N-1) (\lambda'_n(t))^2 g_n(t) + 2 \lambda_n^{N-2}(t) \mathcal{L}_{t,n} Q_t^{(n)}(\psi_n) \\
&= \lambda_n^{N-2}(t) N(N-1) (\lambda'_n(t))^2 g_n(t) + 2 \lambda_n^{N-2}(t) \mathcal{L}_{t,n}(\nu_{t,n}(\psi_n) g_n(t)) \\
&= \lambda_n^{N-2}(t) N(N-1) (\lambda'_n(t))^2 g_n(t) + 2 \lambda_n^{N-2}(t) \nu_{t,n}(\psi_n) \mathcal{L}_{t,n}(g_n(t)) \\
&= \lambda_n^{N-2}(t) N(N-1) (\lambda'_n(t))^2 g_n(t) + 2 \lambda_n^{N-1}(t) \lambda'_n(t) g_n(t).
\end{aligned}$$

In consequence, denoting by  $T_N(t, n)$  the function whose limit (as  $n \rightarrow \infty$ ) is calculated in (11.22), and utilizing (11.23), we get

$$T_N(t, n) = Q_t^{(n)} \mathcal{L}_{t,n}(g_n(t) \ell^2) + \frac{2}{N} \lambda'_n(t) g_n(t)$$

It thus follows from (11.22) that

$$(11.24) \quad \lambda''_n(t) g_n(t) = Q_t^{(n)} \mathcal{L}_{t,n}(g_n(t) \ell^2).$$

Since, by Proposition 5.3, all the operators  $Q_t^{(n)} : \mathcal{B}_\theta \rightarrow \mathcal{B}_\theta$  are positive, and because of this also non-decreasing, and  $g_n(t) = Q_t^{(n)} \mathbb{1}$  is non-negative, the formula (11.24) yields the following.

$$\begin{aligned}
\lambda''_n(t) g_n(t) &\leq \|g_n(t)\|_\infty Q_t^{(n)}(\mathcal{L}_{t,n}(\ell^2)) \leq \|g_n(t)\|_\infty \|\mathcal{L}_{t,n}(\ell^2)\|_\infty Q_t^{(n)} \mathbb{1} \\
&\leq \|g_n(t)\|_\infty \|\mathcal{L}_t(\ell^2)\|_\infty g_n(t) \\
&\leq 2 \|g_n(t)\|_\infty \|\mathcal{L}_s(\ell^2)\|_\infty g_n(t),
\end{aligned}$$

where the last inequality was written for  $t$  sufficiently close to  $s$ . Canceling out  $g_n(t)$  and noticing that by Proposition 11.9 (g),  $\|g_n(t)\|_\infty = \|Q_t^{(n)} \mathbb{1}\|_\infty \leq C$ , we now finally obtain that

$$\lambda''_n(t) \leq 2C \|\mathcal{L}_s(\ell^2)\|_\infty,$$

and the proof is complete. □

Now we shall prove the following.

**Lemma 11.11.** *We have*

- (a) *For every  $n \geq 1$  the function  $(\theta_S, +\infty) \ni t \mapsto \lambda_n(t)$  is decreasing.*

- (b) For every  $s \in (\theta_S, +\infty)$  and for every  $n \geq 1$  large enough there exists  $\delta > 0$  such that the function  $\lambda_n|_{(s-\delta, s+\delta)}$  is strictly decreasing, in fact  $\lambda'_n \leq \frac{1}{4}\lambda'(s)$  on  $(s-\delta, s+\delta)$ .
- (c) For every  $t \in (\theta_S, +\infty)$  and for every  $n \geq 1$ ,  $\lambda_n(t) \leq \lambda(t)$ .
- (d) For every  $n \geq 1$ ,  $\lim_{t \rightarrow +\infty} \lambda_n(t) = 0$ .
- (e) For every  $n \geq 1$  large enough there exists a unique  $b_n > 0$  such that  $\lambda_n(b_n) = 1$ .

*Proof.* For part (a), Proposition 11.9 implies that

$$(11.25) \quad \lambda_n(t) = \lim_{k \rightarrow \infty} \|\mathcal{L}_{t,n}^k \mathbb{1}\|_\infty^{1/k},$$

and since for each  $n \geq 1$  the function  $t \mapsto \|\mathcal{L}_{t,n}^k \mathbb{1}\|_\infty$  is decreasing, item (a) follows immediately. For part (b) note that  $\lambda'(s) < 0$ . Hence, by Theorem 11.1,  $\lambda'_n(s) < \frac{1}{2}\lambda'(s) < 0$  for all  $n \geq 1$  large enough, say  $n \geq N_1$ . Take now  $\delta \in (0, \eta)$  so small that  $M\delta \leq -\frac{1}{4}\lambda'(s)$ , where  $M \geq 0$  is the constant coming from Lemma 11.10. By the Mean Value Theorem  $\lambda'_n(t) = \lambda'_n(s) + \lambda''_n(u)(t-s)$  for every  $t \in (s-\delta, s+\delta)$  and some  $u \in (s-\delta, s+\delta)$  depending on  $t$ . Hence, applying Lemma 11.10, we get for all  $n \geq N_1$  and all  $t \in (s-\delta, s+\delta)$  that

$$\lambda'_n(t) < \frac{1}{2}\lambda'(s) + M\delta < 0.$$

Thus item (b) is proved. Similarly as in item (a), item (c) immediately follows from (11.25) and inequality  $\mathcal{L}_{t,n}^k \mathbb{1} \leq \mathcal{L}_t^k \mathbb{1}$ . Item (d) is an immediate consequence of item (c) and the well-known fact (see [14]) that  $\lim_{t \rightarrow +\infty} \lambda(t) = 0$ . Proving (e), it is well-known (see again [14]) that there exists a unique  $b \in (\theta_S, +\infty)$  such that

$$\lambda(b) = 1.$$

Let  $\delta > 0$  be the value produced in item (b) for  $s = b$ . We know that

$$\lambda(b - \frac{1}{2}\delta) > 0 \quad \text{and} \quad \lambda(b + \frac{1}{2}\delta) < 0.$$

It follows from Proposition 11.9 (f) that

$$\lambda_n(b - \frac{1}{2}\delta) \geq \frac{1}{2}\lambda(b - \frac{1}{2}\delta) > 0 \quad \text{and} \quad \lambda_n(b + \frac{1}{2}\delta) \leq \frac{1}{2}\lambda(b + \frac{1}{2}\delta) < 0$$

for all  $n \geq 1$  large enough, say  $n \geq N_2$ . Because of the choice of  $\delta > 0$  and because of item (b), we may also have  $N_2 \geq 1$  so large that the function  $\lambda_n|_{[b-\frac{1}{2}\delta, b+\frac{1}{2}\delta]}$  is strictly decreasing for every  $n \geq N_2$ . Therefore, for every  $n \geq N_2$  the function  $\lambda_n|_{[b-\frac{1}{2}\delta, b+\frac{1}{2}\delta]}$  has a unique zero. Along with item (a) this finishes the proof of item (e). The proof of Lemma 11.11 is complete.  $\square$

**Remark 11.12.** With the help of Proposition 11.9 we could have strengthened Theorem 11.1 to show uniform convergence with respect to  $t$  ranging over compact subsets of  $(\theta_S, +\infty)$ . However, we really do not need this in the current paper.

By analogy to the unperturbed case, we call the numbers  $b_n$  produced in this lemma Bowen's parameters. Now we can prove the following.

**Proposition 11.13.** *With the settings of the current section (in particular with the stronger condition (U2\*) replacing (U2)), we have*

$$\lim_{n \rightarrow \infty} \frac{b - b_n}{\mu_b(U_n)} = \begin{cases} 1/\chi_{\mu_b} & \text{if (U4A) holds} \\ (1 - |\varphi'_\xi(\pi(\xi^\infty))|)/\chi_{\mu_b} & \text{if (U4B) holds} \end{cases}$$

*Proof.* Since the functions  $(\theta_S, +\infty) \ni t \mapsto \lambda_n(t)$ ,  $n \geq 1$ , are all real-analytic by the Kato-Rellich Perturbation Theorem, making use of Lemma 11.10, we can apply Taylor's Theorem to get

$$1 = \lambda_n(b_n) = \lambda_n(b) + \lambda'(b) + \mathcal{O}((b - b_n)^2).$$

Equivalently,

$$\frac{1 - \lambda_n(b)}{b - b_n} = -\lambda'(b) + \mathcal{O}(b - b_n).$$

Denoting by  $d(\xi)$  the right-hand side of the formula appearing in Proposition 6.1, and using this proposition along with the fact that  $\lambda(b) = 1$ , we thus get

$$\lim_{n \rightarrow \infty} \frac{\mu_b(U_n)}{b - b_n} = -\lambda'(b)d^{-1}(\xi).$$

Equivalently,

$$(11.26) \quad \lim_{n \rightarrow \infty} \frac{b - b_n}{\mu_b(U_n)} = -\frac{1}{\lambda'(b)}d(\xi).$$

But expanding (11.7) with  $n = \infty$ , we get  $\lambda'(b) = -\lambda(b)\chi_{\mu_b} = -\chi_{\mu_b}$ , and inserting this into (11.26) completes the proof.  $\square$

Now we shall link Bowen's parameters  $b_n$  to geometry. We shall prove the following.

**Theorem 11.14.** *Let  $\mathcal{S} = \{\varphi_e\}_{e \in E}$  be a finitely primitive strongly regular conformal graph directed Markov system. Let  $(U_n)_{n=0}^\infty$  be a nested sequence of open subsets of  $E_A^\infty$  satisfying conditions (U0), (U1), and (U2\*) with  $s = b_S$ . If for  $n \geq 0$*

$$\tilde{K}_n := \bigcap_{k=0}^\infty \sigma^{-k}(U_n^c) = \{\omega \in E_A^\infty : \forall_{(k \geq 0)} \sigma^k(\omega) \notin U_n\}$$

and

$$K_n := \pi_S(\tilde{K}_n).$$

Then

$$\text{HD}(K_n) = b_n$$

for all  $n \geq 0$  large enough.

*Proof.* Put

$$h_n := \text{HD}(K_n).$$

We first shall prove that

$$h_n \leq b_n$$

for all  $n \geq 0$  large enough. Assume that  $\delta > 0$  is chosen so small that the conclusion of Lemma 11.11 (b) holds. Take then an arbitrary  $t > b_n$ . Fix any  $q \geq 1$ . Define

$$\tilde{K}_n(q) := \{\omega \in \tilde{K}_n : \omega_n = q \text{ for infinitely many } n\}$$

and

$$K_n(q) := \pi(\tilde{K}_n(q)).$$

Our first goal is to show that

$$(11.27) \quad \text{HD}(K_n(q)) \leq b_n$$

for all  $n \geq 0$  large enough. Indeed, for every  $k \geq 1$  let

$$\tilde{E}_k(q) := \{\omega|_k : \omega \in \tilde{K}_n(q) \text{ and } \omega_{k+1} = q\}.$$

Fix an arbitrary  $\alpha \in E_A^\infty$  such that  $q\alpha \in E_A^\infty$ . Then, using (BDP), Proposition 11.9 (c), (e), and (g), along with Lemma 3.1, we get

$$\begin{aligned} \sum_{\tau \in \tilde{E}_k(q)} \text{diam}^t(\varphi_\tau(X_{t(\tau)})) &= \sum_{\tau \in \tilde{E}_k(q)} \text{diam}^t(\varphi_\tau(X_{t(q)})) \asymp \sum_{\tau \in \tilde{E}_k(q)} \|\varphi'_\tau\|^t \\ &\asymp \sum_{\tau \in \tilde{E}_k(q)} |\varphi'_\tau(\pi(q\alpha))|^t \leq \mathcal{L}_t^k(\mathbb{1}_n^k)(q\alpha) = \mathcal{L}_{t,n}^k(\mathbb{1})(q\alpha) \\ &= \lambda_n^k(t) Q_t^{(n)}(\mathbb{1})(q\alpha) + S^k(\mathbb{1})(q\alpha) \\ &\leq \lambda_n^k(t) \|Q_t^{(n)}(\mathbb{1})\|_\infty + \|S^k(\mathbb{1})\|_\infty \\ &\leq C\lambda_n^k(t) + C\kappa^k \\ &= C(\lambda_n^k(t) + \kappa^k). \end{aligned}$$

Therefore, for every  $k \geq 0$ , using the facts that  $\lambda_n(t) < 1$  (Lemma 11.11 (b)) and that  $\kappa < 1$ , we get

$$(11.28) \quad \sum_{k=l}^{\infty} \sum_{\tau \in \tilde{E}_k(q)} \text{diam}^t(\varphi_\tau(X_{t(\tau)})) \leq C \sum_{k=l}^{\infty} (\lambda_n^k(t) + \kappa^k) \leq C(1 - \lambda_n(t))^{-1} \lambda_n^l(t) + (1 - \kappa)^{-1} \kappa^l.$$

Since  $\bigcup_{k=l}^{\infty} \bigcup_{\tau \in \tilde{E}_k(q)} \varphi_\tau(X_{t(\tau)})$  is a cover of  $K_n(q)$  whose diameters converge (exponentially fast) to zero as  $l \rightarrow \infty$ , formula (11.28) yields  $H_t(K_n(q)) = 0$ . Therefore,  $\text{HD}(K_n(q)) \leq t$ . As  $t > b_n$  was arbitrary, this gives formula (11.27). Let

$$\begin{aligned} \tilde{K}_n(\infty) &:= \{\omega \in \tilde{K}_n : \text{at least one } q \in \mathbb{N} \text{ appears in } \omega \text{ infinitely many times}\} \\ &= \bigcup_{q=1}^{\infty} \tilde{K}_n(q) \end{aligned}$$

and let

$$K_n(\infty) := \pi(\tilde{K}_n(\infty)) = \bigcup_{q=1}^{\infty} K_n(q).$$

Formula (11.27) and  $\sigma$ -stability of Hausdorff dimension then imply that

$$(11.29) \quad \text{HD}(K_n(\infty)) \leq b_n$$

Now, for every integer  $l \geq 1$  let

$$\tilde{K}_n^*(l) := \{\omega \in E_A^\infty : \text{the letters } 1, 2, \dots, l \text{ appear in } \omega \text{ only finitely many times}\}$$

and

$$\tilde{K}_n^0(l) := \{\omega \in E_A^\infty : \text{the letters } 1, 2, \dots, l \text{ do not appear in } \omega \text{ at all}\}$$

Furthermore,

$$K_n^*(l) := \pi(\tilde{K}_n^*(l)) \text{ and } K_n^0(l) := \pi(\tilde{K}_n^0(l)).$$

But

$$K_n^*(l) \subseteq \bigcup_{\omega \in E_A^*} \varphi_\omega(K_n^0(l)),$$

and therefore

$$\text{HD}(K_n^*(l)) = \text{HD}(K_n^0(l)).$$

But  $K_n \setminus K_n(\infty) \subseteq \bigcap_{l=1}^\infty K_n^*(l)$ . Hence, applying Theorem 4.3.6 in [14], we get

$$\text{HD}(K_n \setminus K_n(\infty)) \leq \inf_{l \geq 1} \{\text{HD}(K_n^*(l))\} = \inf_{l \geq 1} \{\text{HD}(K_n^0(l))\} = \theta_S < b_S (= b).$$

Since  $\lim_{n \rightarrow \infty} b_n = b$ , this implies that for all  $n \geq 1$  large enough  $\text{HD}(K_n \setminus K_n(\infty)) < b_n$ . Along with (11.29) this yields

$$(11.30) \quad \text{HD}(K_n) \leq b_n.$$

The opposite inequality is even more involved. The main difficulty is caused by the fact that the Variational Principle (for topological pressure) holds in the case of infinite alphabet only for topologically mixing subshifts. Our idea is, given  $n \geq 0$ , to make smaller and smaller perturbations of the operators  $\mathcal{L}_n$ , such that the difference is an operator acting essentially on a finite alphabet symbol space. So, given an integer  $l \geq 1$  we denote

$$\mathbb{N}_l := \{1, 2, \dots, l\}.$$

Given also  $n \geq 0$  we set

$$U_n^{(l)} := U_n \cup \mathbb{N}_l^c.$$

For a time being  $\varphi : E_A^\infty \rightarrow \mathbb{R}$  is again an arbitrary summable Hölder continuous potential and  $\mu_\varphi$  is the corresponding  $\sigma$ -invariant Gibbs/equilibrium state. Later on we will need  $\varphi$  to be of the form  $\varphi_t(\omega) = t \log |\varphi'_{\omega_0}(\pi(\sigma(\omega)))|$ . Given  $q \geq n$  let  $l_q \geq 1$  be the least integer such that

$$(11.31) \quad \mu_\varphi(\mathbb{N}_{l_q}^c) \leq \rho^n.$$

Set

$$U_n(q) := U_n^{l_q}.$$

Of course each open set  $U_n(q)$  is a disjoint union of cylinders of length  $q$  so that condition (U1) is satisfied for the sequence  $(U_n(q))_{q=n}^\infty$ .  $\mathcal{L} := \mathcal{L}_\varphi$  is the fully normalized transfer operator associated to  $\varphi$ . As in Section 4 we define the operators

$$\mathcal{L}_{n,q}(g) := \mathcal{L}(\mathbb{1}_{U_n^c(q)}g).$$

The space  $\mathcal{B}_\theta$  and the norm  $\|\cdot\|_\theta$  remain unchanged. We however naturally adjust the seminorm  $|\cdot|_*$  to depend on our sequence  $(U_n(q))_{q=n}^\infty$ . We set for  $g \in \mathcal{B}$ :

$$|g|_n^* := \sup_{j \geq 0} \sup_{m \geq 1} \left\{ \theta^{-m} \int_{\sigma^{-j}(\mathbb{N}_{l_m}^c)} |g| d\mu_\varphi \right\}$$

and

$$\|g\|_n^* := \|g\|_1 + |g|_n^*.$$

We intend to apply Keller and Liverani (see [11]) perturbation results. Because of (11.31), Lemma 4.1 goes through for the norm  $\|\cdot\|_n^*$ . We put

$$\begin{aligned} \mathbb{1}_{n,q}^k &:= \prod_{j=0}^{k-1} \mathbb{1}_{\sigma^{-j}(U_n^c(q))} = \prod_{j=0}^{k-1} \mathbb{1}_{U_n^c(q)} \circ \sigma^j, \\ \mathbb{1}_{n,q}^{k,*} &:= \prod_{j=0}^{k-1} \mathbb{1}_{\sigma^{-j}(\mathbb{N}_{l_q})} = \prod_{j=0}^{k-1} \mathbb{1}_{\mathbb{N}_{l_q}} \circ \sigma^j, \end{aligned}$$

and note that

$$\mathbb{1}_{n,q}^k = \mathbb{1}_n^k \cdot \mathbb{1}_{n,q}^{k,*}.$$

The proof of Lemma 4.2 goes the same way for the operators  $\mathcal{L}_{n,q}^k$  with only formal change of  $\mathbb{1}_n^k$  by  $\mathbb{1}_{n,q}^k$  and  $U_m$  by  $U_m(q)$ . It gives:

**Lemma 11.15.** *For every  $k \geq 1$  and for every  $q \geq n$ , we have that*

$$\|\mathcal{L}_{n,q}^k\|_n^* \leq 1.$$

Lemmas 4.3, 4.4, and Corollary 4.5 used only the (U1) property of the sequence  $(U_n)_{n=0}^\infty$ , and therefore these apply to the sets  $U_n(q)$ ,  $q \geq n$ , and the operators  $\mathcal{L}_{n,q}^k$  (to be clear, the role of  $n$  in these three results is now played by the pair  $(n, q)$ ). Fix  $a, b > 1$  such that  $\frac{1}{a} + \frac{1}{b} = 1$ . We shall prove the following analogue of Lemma 5.1.

**Lemma 11.16.** *For every  $n \geq 0$  we have*

$$\|\mathcal{L}_n - \mathcal{L}_{n,q}\| \leq 2(\rho^{1/b})^q.$$

*Proof.* Fix an arbitrary  $g \in \mathcal{B}_\theta$  with  $\|g\|_\theta \leq 1$ . Using Lemma 3.1 and (11.31), we get

$$\begin{aligned} \|(\mathcal{L}_n - \mathcal{L}_{n,q})g\|_1 &= \|\mathcal{L}(\mathbb{1}_{U_n^c \setminus U_n^c(q)}g)\|_1 = \|\mathbb{1}_{U_n^c \setminus U_n^c(q)}g\|_1 \leq \mu_\varphi(U_n^c \setminus U_n^c(q))\|g\|_\infty \\ &= \mu_\varphi(U_n^c \cap U_n(q))\|g\|_\infty = \mu_\varphi(U_n^c \cap (U_n \cup \mathbb{N}_{l_q}^c))\|g\|_\infty \\ (11.32) \quad &= \mu_\varphi(U_n^c \cap \mathbb{N}_{l_q}^c)\|g\|_\infty \leq \mu_\varphi(\mathbb{N}_{l_q}^c)\|g\|_\theta \\ &\leq \rho^q\|g\|_\theta \leq \rho^q \leq (\rho^{1/b})^q. \end{aligned}$$

Also, using Cauchy-Schwarz Inequality, we get

$$\begin{aligned}
\theta^{-m} \int_{\sigma^{-j}(\mathbb{N}_{l_m}^c)} |(\mathcal{L}_n - \mathcal{L}_{n,q})g| d\mu_\varphi &= \\
&= \theta^{-m} \int_{E_A^\infty} \mathbb{1}_{\mathbb{N}_{l_m}^c} \circ \sigma^j |\mathcal{L}(\mathbb{1}_{U_n^c \setminus U_n^c(q)}g)| d\mu_\varphi \\
&\leq \theta^{-m} \|g\|_\infty \int_{E_A^\infty} \mathbb{1}_{\mathbb{N}_{l_m}^c} \circ \sigma^j \mathcal{L}(\mathbb{1}_{U_n^c \setminus U_n^c(q)}) d\mu_\varphi \\
&= \theta^{-m} \|g\|_\infty \int_{E_A^\infty} \mathcal{L}(\mathbb{1}_{\mathbb{N}_{l_m}^c} \circ \sigma^{j+1} \mathbb{1}_{U_n^c \setminus U_n^c(q)}) d\mu_\varphi \\
&= \theta^{-m} \|g\|_\infty \int_{E_A^\infty} \mathbb{1}_{\mathbb{N}_{l_m}^c} \circ \sigma^{j+1} \mathbb{1}_{U_n^c \setminus U_n^c(q)} d\mu_\varphi \\
&= \theta^{-m} \|g\|_\infty \int_{E_A^\infty} \mathbb{1}_{\mathbb{N}_{l_m}^c} \circ \sigma^{j+1} \mathbb{1}_{\mathbb{N}_{l_q}^c} d\mu_\varphi \\
&= \theta^{-m} \|g\|_\infty \int_{E_A^\infty} \mathbb{1}_{\sigma^{-(j+1)}(\mathbb{N}_{l_m}^c)} \mathbb{1}_{\mathbb{N}_{l_q}^c} d\mu_\varphi \\
&\leq \theta^{-m} \|g\|_\infty \mu_\varphi^{1/a}(\mathbb{N}_{l_m}^c) \mu_\varphi^{1/b}(\mathbb{N}_{l_q}^c) \\
&\leq \theta^{-m} \|g\|_\infty (\rho^{1/a}/\theta)^m \rho^{q/b} \leq \rho^{q/b} \|g\|_\theta \leq \rho^{q/b}.
\end{aligned}$$

Therefore,  $|(\mathcal{L}_n - \mathcal{L}_{n,q})g|_n^* \leq \rho^{q/b}$ , and together with (11.32), this completes the proof of our lemma.  $\square$

Having all of this, particularly the last lemma, and taking into account the considerations between the end of the proof of Lemma 5.1 and Proposition 5.2, we get the following analogue of the latter for the  $\mathcal{L}_n$  replaced by  $\mathcal{L}_{n,q}$ .

**Lemma 11.17.** *For all integers  $n \geq 0$  large enough and for all  $q \geq n$  large enough there exist two bounded linear operators  $Q_{n,q}, \Delta_{n,q} : \mathcal{B}_\theta \rightarrow \mathcal{B}_\theta$  and complex numbers  $\lambda_{n,q} \neq 0$  with the following properties:*

- (a)  $\lambda_{n,q}$  is a simple eigenvalue of the operator  $\mathcal{L}_{n,q} : \mathcal{B}_\theta \rightarrow \mathcal{B}_\theta$ .
- (b)  $Q_{n,q} : \mathcal{B}_\theta \rightarrow \mathcal{B}_\theta$  is a projector ( $Q_{n,q}^2 = Q_{n,q}$ ) onto the 1-dimensional eigenspace of  $\lambda_{n,q}$ .
- (c)  $\mathcal{L}_{n,q} = \lambda_{n,q} Q_{n,q} + \Delta_{n,q}$ .
- (d)  $Q_{n,q} \circ \Delta_{n,q} = \Delta_{n,q} \circ Q_{n,q} = 0$ .
- (e) There exist  $\kappa_n \in (0, 1)$  and  $C > 0$  such that

$$\|\Delta_{n,q}^k\|_\theta \leq C \kappa_n^k.$$

In particular,

$$\|\Delta_{n,q}^k g\|_\infty \leq \|\Delta_{n,q}^k g\|_\theta \leq C \kappa_n^k \|g\|_\theta$$



for all  $g \in \mathcal{B}_\theta$ .

(f)  $\lim_{q \rightarrow \infty} \lambda_{n,q} = \lambda_n$ .

(g) Enlarging the above constant  $C > 0$  if necessary, we have

$$\|Q_{n,q}\|_\theta \leq C.$$

In particular,

$$\|Q_{n,q}g\|_\infty \leq \|Q_{n,q}g\|_\theta \leq C\|g\|_\theta$$

for all  $g \in \mathcal{B}_\theta$ .

(h)  $\lim_{q \rightarrow \infty} \|Q_{n,q} - Q_n\| = 0$ .

The following lemma can be proved in exactly the same way as was Proposition 5.3.

**Lemma 11.18.** *All eigenvalues  $\lambda_{n,q}$  produced in Lemma 11.17 are real and positive, and all operators  $Q_{n,q} : \mathcal{B}_\theta \rightarrow \mathcal{B}_\theta$  preserve  $\mathcal{B}_\theta(\mathbb{R})$  and  $\mathcal{B}_\theta^+(\mathbb{R})$ , the subspaces of  $\mathcal{B}_\theta$  consisting, respectively, of real-valued functions and positive real-valued functions.*

**Remark 11.19.** How large  $n$  needs to be in Lemmas 11.17 and 11.18 is determined by the requirement that the assertions of Proposition 5.2 hold for such  $n$ .

Now, let us consider the dynamical systems  $\sigma : \tilde{K}_n(q) \rightarrow \tilde{K}_n(q)$ , where

$$\tilde{K}_n(q) := \bigcap_{k=0}^{\infty} \sigma^{-k}(U_n^c(q)) \quad \text{and} \quad K_n(q) := \pi(\tilde{K}_n(q)).$$

We shall prove the following.

**Lemma 11.20.** *If  $n \geq 0$  is so large as required in Lemma 11.17, then for all  $q \geq n$  large enough we have that*

$$P(\sigma|_{\tilde{K}_n(q)}, \varphi|_{\tilde{K}_n(q)}) \geq \log \lambda_{n,q}.$$

*Proof.* A straightforward elementary calculation shows that if  $f, g \in \mathcal{B}_\theta$ , then  $\|fg\|_\theta \leq 3\|f\|_\theta\|g\|_\theta$ ; hence in particular  $fg \in \mathcal{B}_\theta$ . This allows us to define a linear functional  $\mu_{n,q} : \mathcal{B}_\theta \rightarrow \mathbb{R}$  by the requirement that

$$Q_{n,q}(gg_{n,q}) = \mu_{n,q}(g)g_{n,q}.$$

Since, by Lemma 11.18,  $Q_{n,q}$  is a positive ( $Q_{n,q}(\mathcal{B}_\theta^+(\mathbb{R})) \subseteq \mathcal{B}_\theta^+(\mathbb{R})$ ) operator and  $Q_{n,q} \neq 0$  all  $q \geq n$  large enough, it follows that  $\mu_{n,q}$  is a positive ( $\mu_{n,q}(\mathcal{B}_\theta^+(\mathbb{R})) \subseteq [0, +\infty)$ ) functional and

$$(11.33) \quad \mu_{n,q}(\mathbb{1}) = 1.$$

Positivity of  $\mu_{n,q}$  immediately implies its monotonicity in the sense that if  $f, g \in \mathcal{B}_\theta$  and  $f(x) \leq g(x)$   $\mu_\varphi$ -a.e. in  $E_A^\infty$ , then

$$(11.34) \quad \mu_{n,q}(f) \leq \mu_{n,q}(g)$$

Now, let  $C_b^u(E_A^\infty)$  be the vector subspace of  $C_b(E_A^\infty)$  consisting of all functions that are uniformly continuous with respect to the metric  $d_\theta$ . Let us define a function  $\mu : C_b^u(E_A^\infty) \rightarrow [0, +\infty)$  by the following formula:

$$(11.35) \quad \mu(g) := \sup \{ \mu_{n,q}(f) : f \leq g \text{ and } f \in H_\theta^b(A) \}.$$

Of course by (11.34) we get that

$$(11.36) \quad \mu|_{H_\theta^b(A)} = \mu_{n,q}|_{H_\theta^b(A)}.$$

Given  $g \in C_b^u(E_A^\infty)$  and  $k \geq 1$  define two functions

$$\underline{g}_k(\omega) := \inf \{ g(\tau) : \tau \in [\omega|k] \} \text{ and } \overline{g}_k(\omega) := \sup \{ g(\tau) : \tau \in [\omega|k] \}.$$

Of course

$$\underline{g}_k \leq g \leq \overline{g}_k$$

and

$$(11.37) \quad \lim_{k \rightarrow \infty} \|g - \underline{g}_k\|_\infty = \lim_{k \rightarrow \infty} \|g - \overline{g}_k\|_\infty = 0.$$

We shall prove that for every  $g \in C_b^u(E_A^\infty)$  we have that

$$(11.38) \quad \mu(g) = \overline{\mu}(g) := \inf \{ \mu_{n,q}(f) : f \geq g \text{ and } f \in H_\theta^b(A) \}.$$

Then for every  $k \geq 1$  we have that

$$\begin{aligned} \mu(g) &\leq \mu_{n,q}(\overline{g}_k) = \mu_{n,q}(\underline{g}_k + (\overline{g}_k - \underline{g}_k)) = \mu_{n,q}(\underline{g}_k) + \mu_{n,q}(\overline{g}_k - \underline{g}_k) \\ &\leq \mu(g) + \mu(\|\overline{g}_k - \underline{g}_k\|_\infty) \\ &= \mu(g) + \|\overline{g}_k - \underline{g}_k\|_\infty, \end{aligned}$$

and invoking (11.37), we obtain  $\mu(g) \leq \overline{\mu}(g) \leq \mu(g)$ , completing the proof of (11.38). We now can prove the following.

**Lemma 11.21.** *The function  $\mu : C_b^u(E_A^\infty) \rightarrow \mathbb{R}$  is a positive linear functional such that  $\mu(\mathbb{1}) = 1$  and  $\mu|_{H_\theta^b(A)} = \mu_{n,q}|_{H_\theta^b(A)}$ .*

*Proof.* Positivity is immediate from formula (11.35). It is also immediate from this formula that

$$(11.39) \quad \mu(\alpha g) = \alpha \mu(g)$$

for every  $\alpha \geq 0$ . Employing also (11.38), we get that

$$\begin{aligned} \mu(-g) &= \inf \{ \mu_{n,q}(f) : f \geq -g \text{ and } f \in H_\theta^b(A) \} \\ &= \inf \{ -\mu_{n,q}(-f) : -f \leq g \text{ and } f \in H_\theta^b(A) \} \\ &= \inf \{ -\mu_{n,q}(f) : f \leq g \text{ and } f \in H_\theta^b(A) \} \\ &= -\sup \{ \mu_{n,q}(f) : f \leq g \text{ and } f \in H_\theta^b(A) \} \\ &= -\mu(g). \end{aligned}$$

Along with (11.40) this implies that

$$(11.40) \quad \mu(\alpha g) = \alpha \mu(g)$$

for every  $g \in C_b^u(E_A^\infty)$  and all  $\alpha \in \mathbb{R}$ . Now fix two functions  $f, g \in C_b^u(E_A^\infty)$ . Because of (11.38) and (11.35) there exist four sequences  $(f_k^-)_1^\infty$ ,  $(f_k^+)_1^\infty$ ,  $(g_k^-)_1^\infty$ , and  $(g_k^+)_1^\infty$  of elements of  $H_\theta^b(A)$  such that

$$f_k^- \leq f \leq f_k^+, \quad g_k^- \leq g \leq g_k^+,$$

and

$$\lim_{k \rightarrow \infty} \mu_{n,q}(f_k^-) = \lim_{k \rightarrow \infty} \mu_{n,q}(f_k^+) = \mu(f) \quad \text{and} \quad \lim_{k \rightarrow \infty} \mu_{n,q}(g_k^-) = \lim_{k \rightarrow \infty} \mu_{n,q}(g_k^+) = \mu(g).$$

Therefore, applying again (11.38) and (11.35), we obtain

$$\mu(f + g) \geq \overline{\lim}_{k \rightarrow \infty} \mu_{n,q}(f_k^- + g_k^-) = \lim_{k \rightarrow \infty} \mu_{n,q}(f_k^-) + \lim_{k \rightarrow \infty} \mu_{n,q}(g_k^-) = \mu(f) + \mu(g)$$

and

$$\mu(f + g) \leq \underline{\lim}_{k \rightarrow \infty} \mu_{n,q}(f_k^+ + g_k^+) = \lim_{k \rightarrow \infty} \mu_{n,q}(f_k^+) + \lim_{k \rightarrow \infty} \mu_{n,q}(g_k^+) = \mu(f) + \mu(g).$$

Hence,

$$\mu(f + g) = \mu(f) + \mu(g),$$

and along with (11.40) this finishes the proof of Lemma 11.21 (the last two assertions of this lemma are immediate consequences of (11.33) and (11.36)).  $\square$

Now we shall prove the following auxiliary fact.

**Lemma 11.22.** *If  $g \in C_b^u(E_A^\infty)$  and  $g|_{\tilde{K}_{n,q}} = 0$ , then  $\mu(g) = 0$ .*

*Proof.* Let

$$\mathcal{F}_{n,q} = \{\omega \in E_A^n : [\omega] \subseteq U_{n,q}^c\},$$

and note that  $\mathcal{F}_{n,q}$  is a finite set. For every  $k \geq 1$  let

$$U_{n,k}^{ck} := \bigcap_{j=0}^{k-1} \sigma^{-j}(U_n^c).$$

We shall prove the following.

**Claim 1:** There exists  $p \geq 1$  such that if  $\omega \in E_A^{kn}$  and  $[\omega] \subseteq U_{n,k}^{ck}$ , then  $[\omega|_{kn-pn}] \cap \tilde{K}_{n,q} \neq \emptyset$ .

*Proof.* Let

$$(11.41) \quad p := \#\mathcal{F}_{n,q} + 1 < +\infty.$$

Seeking a contradiction suppose that  $k > p$  and

$$(11.42) \quad [\omega|_{(k-p)n}] \cap \tilde{K}_{n,q} = \emptyset$$

for some  $\omega \in E_A^{kn}$  with  $[\omega] \subseteq U_{n,q}^{ck}$ . Because  $|\omega|_{(k-p)n+1}^{kn} = pn$  and because  $\omega|_{(k-p)n+1}^{kn}$  is a concatenation of non-overlapping blocks from  $\mathcal{F}_{n,q}$ , it follows from (11.41) that there are two non-overlapping subblocks of  $\omega|_{(k-p)n+1}^{kn}$  forming the same element of  $\mathcal{F}_{n,q}$ . Let

$\omega|_{(l-1)n+1}^{ln}$ ,  $k-p \leq l-1 \leq k-1$  be the latter of these two blocks, and let the former, denote it by  $\tau$ , have the last coordinate  $j$  ( $j \leq (l-1)n$ ). But then the infinite word  $\omega|_{ln}(\omega|_{j+1}^{ln})^\infty$  is an element of  $E_A^\infty$  and  $\omega|_{ln}(\omega|_{j+1}^{ln})^\infty$  is a concatenation of non-overlapping blocks from  $\mathcal{F}_{n,q}$ . But this means that  $\omega|_{ln}(\omega|_{j+1}^{ln})^\infty \in \tilde{K}_{n,q}$ . Thus,  $[\omega|_{ln}] \cap \tilde{K}_{n,q} \neq \emptyset$ . As  $l \geq k-p$ , this contradicts (11.42) and finishes the proof of Claim 1.  $\square$

Now passing to the direct proof of our lemma, fix  $\varepsilon > 0$  arbitrary. Since  $g|_{\tilde{K}_{n,q}} = 0$  and  $g \in C_b^u(E_A^\infty)$ , there exists  $l \geq 1$  sufficiently large that

$$(11.43) \quad \|g\|_{[\omega]} \leq \varepsilon/2$$

if  $|\omega| \geq l$  ( $\omega \in E_A^l$ ) and  $[\omega] \cap \tilde{K}_{n,q} \neq \emptyset$ . Take any  $k \geq l+p$  so large that  $\|\bar{g}_{kn} - g\|_\infty \leq \varepsilon/2$ . Employing Claim 1, (11.43), Lemma 11.21, and (11.36), we get

$$\begin{aligned} \mu(g)g_{n,q} &\leq \mu(\bar{g}_{kn})g_{n,q} = \mu_{n,q}(\bar{g}_{kn})g_{n,q} = Q_{n,q}(\bar{g}_{kn}g_{n,q}) = \lambda_{n,q}^{-kn} \mathcal{L}_{n,q}^{kn} Q_{n,q}(\bar{g}_{kn}g_{n,q}) \\ &= \lambda_{n,q}^{-kn} Q_{n,q} \mathcal{L}_{n,q}^{kn} (\bar{g}_{kn}g_{n,q}) \\ &= \lambda_{n,q}^{-kn} Q_{n,q} \left( \tau \mapsto \sum_{[\omega] \subseteq U_{n,q}^{ck}: A\omega_{kn}\tau_0=1} \bar{g}_{kn}(\omega\tau)g_{n,q}(\omega\tau)e^{\varphi_{kn}(\omega\tau)} \right) \\ &\leq \lambda_{n,q}^{-kn} Q_{n,q} \left( \tau \mapsto \varepsilon \sum_{A\omega_{kn}\tau_0=1} \mathbb{1}_n^{kn}(\omega\tau)g_{n,q}(\omega\tau)e^{\varphi_{kn}(\omega\tau)} \right) \\ &= \varepsilon \lambda_{n,q}^{-kn} Q_{n,q} \mathcal{L}_{n,q}^{kn} (g_{n,q}) = \varepsilon Q_{n,q}(g_{n,q}) \leq \varepsilon \|g_{n,q}\|_\infty Q_{n,q}(\mathbb{1}) \\ &= \varepsilon \|g_{n,q}\|_\infty g_{n,q} \leq \varepsilon \|g_{n,q}\|_\theta g_{n,q}. \end{aligned}$$

Hence,

$$\mu(g) \leq \|g_{n,q}\|_\theta \varepsilon.$$

Likewise,  $-\mu(g) = \mu(-g) \leq \|g_{n,q}\|_\theta \varepsilon$ , and in consequence.

$$|\mu(g)| \leq \|g_{n,q}\|_\theta \varepsilon.$$

Letting  $\varepsilon \searrow 0$  we thus get that  $\mu(g) = 0$  finishing the proof of Lemma 11.22  $\square$

Since every function  $g \in C(\tilde{K}_{n,q})$  is uniformly continuous, it extends to some uniformly continuous function  $\tilde{g} : E_A^\infty \rightarrow \mathbb{R}$ . The value

$$\mu(g) := \mu(\tilde{g})$$

is then, by virtue of, Lemma 11.22, independent of the choice of extension  $\tilde{g} \in C_b^u(E_A^\infty)$  of  $g$ . By Lemma 11.21., we get the following.

**Lemma 11.23.** *The function  $\tilde{\mu} : C(\tilde{K}_{n,q}) \rightarrow \mathbb{R}$  is a positive linear functional such that  $\tilde{\mu}(\mathbb{1}) = 1$ . Thus by the Riesz Representation Theorem  $\tilde{\mu}$  represents a Borel probability measure on  $\tilde{K}_{n,q}$ .*

We shall prove the following.

**Lemma 11.24.** *The measure  $\tilde{\mu}$  on  $\tilde{K}_{n,q}$  is  $\sigma$ -invariant.*

*Proof.* Let  $g \in C(\tilde{K}_{n,q})$ . Let  $\tilde{g} \in C_b^u(E_A^\infty)$  be an extension of  $g$ . Then  $\tilde{g} \circ \sigma \in C_b^u(E_A^\infty)$  and it extends  $g \circ \sigma$ . Fix  $\varepsilon > 0$  and take  $\tilde{g}_+$  and  $\tilde{g}_-$  both in  $H_\theta^b(A)$ , such that  $\tilde{g}_- \leq \tilde{g} \leq \tilde{g}_+$  and

$$\mu_{n,q}(\tilde{g}_+) - \varepsilon \leq \mu(\tilde{g}) \leq \mu_{n,q}(\tilde{g}_-) + \varepsilon.$$

Of course then we also have  $\tilde{g}_+ \circ \sigma, \tilde{g}_- \circ \sigma \in H_\theta^b(A)$  and  $\tilde{g}_- \circ \sigma \leq \tilde{g} \circ \sigma \leq \tilde{g}_+ \circ \sigma$ . We thus get

$$\begin{aligned} \tilde{\mu}(g \circ \sigma)g_{n,q} &= \mu(\tilde{g} \circ \sigma)g_{n,q} \leq \mu(\tilde{g}_+ \circ \sigma)g_{n,q} = Q_{n,q}(g_{n,q}\tilde{g}_+ \circ \sigma) \\ &= \lambda_{n,q}^{-kn} \mathcal{L}_{n,q}^{kn} Q_{n,q}(g_{n,q}\tilde{g}_+ \circ \sigma) \\ &= \lambda_{n,q}^{-kn} Q_{n,q} \mathcal{L}_{n,q}^{kn}(g_{n,q}\tilde{g}_+ \circ \sigma) \\ &= \lambda_{n,q}^{-kn} Q_{n,q}(\tilde{g}_+ \mathcal{L}_{n,q}^{kn}(g_{n,q})) \\ &= Q_{n,q}(\tilde{g}_+ g_{n,q}) \\ &= \mu_{n,q}(\tilde{g}_+)g_{n,q} \\ &\leq (\mu(\tilde{g}) + \varepsilon)g_{n,q} = (\mu(\tilde{g}) + \varepsilon)g_{n,q}. \end{aligned}$$

Hence,  $\tilde{\mu}(g \circ \sigma) \leq \mu(\tilde{g}) + \varepsilon$ . By letting  $\varepsilon \searrow 0$  this yields  $\tilde{\mu}(g \circ \sigma) \leq \mu(\tilde{g})$ . Likewise, working with  $\tilde{g}_-$  instead of  $\tilde{g}_+$ , we get  $\tilde{\mu}(g \circ \sigma) \geq \mu(\tilde{g})$ . Thus  $\tilde{\mu}(g \circ \sigma) = \mu(\tilde{g})$  and the proof is complete.  $\square$

We now pass to the direct proof of the inequality being the assertion of Lemma 11.20. Given any  $\omega \in \tilde{K}_{n,q}$ , we have for every  $k \geq 1$  that

$$\begin{aligned} \tilde{\mu}(\mathbb{1}_{[\omega|_{kn}]})g_{n,q} &= \mu_{n,q}(\mathbb{1}_{[\omega|_{kn}]})g_{n,q} = Q_{n,q}(\mathbb{1}_{[\omega|_{kn}]}g_{n,q}) \\ &= \lambda_{n,q}^{-kn} \mathcal{L}_{n,q}^{kn} Q_{n,q}(\mathbb{1}_{[\omega|_{kn}]}g_{n,q}) = \lambda_{n,q}^{-kn} Q_{n,q} \mathcal{L}_{n,q}^{kn}(\mathbb{1}_{[\omega|_{kn}]}g_{n,q}) \\ &= \lambda_{n,q}^{-kn} Q_{n,q}(\tau \mapsto e^{\varphi_{kn}(\omega|_{kn}\tau)} g_{n,q}(\omega|_{kn}\tau)). \end{aligned}$$

Now, because of Lemma 2.2,  $M_\varphi^{-1}e^{\varphi_{kn}(\omega|_{kn}\tau)} \leq e^{\varphi_{kn}(\omega)} \leq M_\varphi e^{\varphi_{kn}(\omega|_{kn}\tau)}$ . Therefore, the monotonicity of  $Q_{n,q}$  yields

$$\begin{aligned} M_\varphi^{-1} \lambda_{n,q}^{-kn} e^{\varphi_{kn}(\omega)} Q_{n,q}(\tau \mapsto g_{n,q}(\omega|_{kn}\tau)) \\ \leq \tilde{\mu}(\mathbb{1}_{[\omega|_{kn}]}g_{n,q}) \\ \leq M_\varphi \lambda_{n,q}^{-kn} e^{\varphi_{kn}(\omega)} Q_{n,q}(\tau \mapsto g_{n,q}(\omega|_{kn}\tau)). \end{aligned}$$

We are only interested in the right-hand side of this inequality. We further have

$$\tilde{\mu}(\mathbb{1}_{[\omega|_{kn}]}g_{n,q})g_{n,q} \leq M_\varphi \|g_{n,q}\|_\infty \lambda_{n,q}^{-kn} e^{\varphi_{kn}(\omega)} Q_{n,q}(\mathbb{1}) = M_\varphi \|g_{n,q}\|_\infty \lambda_{n,q}^{-kn} e^{\varphi_{kn}(\omega)} g_{n,q}.$$

Hence

$$\tilde{\mu}(\mathbb{1}_{[\omega|_{kn}]}g_{n,q}) \leq M_\varphi \|g_{n,q}\|_\infty \lambda_{n,q}^{-kn} e^{\varphi_{kn}(\omega)}.$$

Denoting by  $\alpha$  the partition of  $E_A^\infty$  into cylinders of length one, i. e. the partition  $\{[e]\}_{e \in E}$ , we therefore get

$$\begin{aligned} H_{\tilde{\mu}}(\alpha^{kn}) + kn\tilde{\mu}(\varphi) &= H_{\tilde{\mu}}(\alpha^k) + \tilde{\mu}(\varphi_{kn}) = \int_{\tilde{K}_{n,q}} -\log \tilde{\mu}([\omega|_k]) d\tilde{\mu}(\omega) + \tilde{\mu}(\varphi_{kn}) \\ &\geq -\log(M_\varphi \|g_{n,q}\|_\infty) + kn \log \lambda_{n,q} - \tilde{\mu}(\varphi_{kn}) + \tilde{\mu}(\varphi_{kn}) \\ &= -\log(M_\varphi \|g_{n,q}\|_\infty) + kn \log \lambda_{n,q}. \end{aligned}$$

Since  $\alpha|_{\tilde{K}_{n,q}}$  is a finite generating partition, we thus get that

$$h_{\tilde{\mu}}(\sigma) + \tilde{\mu}(\varphi) = \lim_{k \rightarrow \infty} \frac{1}{k} (H_{\tilde{\mu}}(\alpha^{kn}) + kn\tilde{\mu}(\varphi)) \geq \log \lambda_{n,q}.$$

Hence, invoking the Variational Principle we get

$$P(\sigma|_{\tilde{K}_n(q)}, \varphi|_{\tilde{K}_n(q)}) \geq \log \lambda_{n,q}.$$

and the proof of Lemma 11.20 is complete.  $\square$

Aiming now directly towards proving the inequality

$$(11.44) \quad \text{HD}(K_n) \geq b_n$$

for all  $n \geq 1$  large enough, we apply Proposition 11.9 with  $s = b (= b_S)$ . Let  $n_b \geq 1$  be the integer produced in this proposition, let  $\delta > 0$  be the minimum of both  $\delta_s$ , the one produced in Proposition 11.9 and the one coming from Lemma 11.11 (b). Let  $N_b \geq n_b$  be so large (depending only on  $s = b$ ) that the assertions of Lemma 11.11 are true for all  $n \geq N_b$ . By Proposition 11.13 there exists  $N_b^* \geq N_b$  so large that  $b_n \in (b - \delta, b + \delta)$  for all  $n \geq N_b^*$ . Take an arbitrary integer  $n$  with this property, i. e.  $n \geq N_b^*$ . Fix any  $t \in (b - \delta, b_n)$ . By Lemma 11.11 (b) and (e), we have that  $\lambda_n(t) > 1$ . Since  $n \geq N_b^* \geq n_b$ , it follows from Proposition 11.9 that the assertions of Proposition 5.2 hold for this  $n$ . In turn, it therefore follows from Remark 11.19 that Lemma 11.17 holds for this  $n$ . Its item (f) yields some  $q \geq n$  such that  $\lambda_{n,q} > 1$ . By virtue of Lemma 11.20 and the Variational Principle, for all  $q \geq n$  large enough there exists a Borel probability  $\sigma$ -invariant measure  $\mu$  on  $\tilde{K}_n(q)$  such that  $h_\mu(\sigma) - t\chi_\mu > 0$ . But then invoking Theorem 4.4.2 from [14], we get that

$$\text{HD}(K_n) \geq \text{HD}(K_{n,q}) \geq \text{HD}(\mu \circ \pi^{-1}) = \frac{h_\mu(\sigma)}{\chi_\mu} > t.$$

So, letting  $t \nearrow b_n$ , inequality (11.44) follows. Along with (11.27) this completes the proof of Theorem 11.14.  $\square$

As a direct consequence of this theorem and Proposition 11.13, we get the following.

**Proposition 11.25.** *With the hypotheses of Theorem 11.14 we have that*

$$(11.45) \quad \lim_{n \rightarrow \infty} \frac{\text{HD}(J_S) - \text{HD}(K_n)}{\mu_b(U_n)} = \begin{cases} 1/\chi_{\mu_b} & \text{if (U4A) holds} \\ (1 - |\varphi'_\xi(\pi(\xi^\infty))|)/\chi_{\mu_b} & \text{if (U4B) holds.} \end{cases}$$

## 12. ESCAPE RATES FOR CONFORMAL GDMSs; HAUSDORFF DIMENSION

This mini-section is the main fruit of the labor in the previous section. It pertains to the rate of decay of Hausdorff dimension of escaping points. It contains, in particular, Theorem 12.1, the second main result of this manuscript. Given  $z \in E_A^\infty$  and  $r > 0$  let

$$K_z(r) := \pi(\tilde{K}_z(r)),$$

where

$$\tilde{K}_z(r) := \left\{ \omega \in E_A^\infty : \forall_{n \geq 0} \sigma^n(\omega) \notin \pi^{-1}(B(\pi(z), r)) \right\} = \bigcap_{n=0}^{\infty} \sigma^{-n}(\pi^{-1}(B(\pi(z), r))).$$

We say that a parameter  $t > \theta_S$  is powering at a point  $\xi \in X$  if there exist  $\alpha > 0$ ,  $C > 0$ , and  $\delta > 0$  such that

$$(12.1) \quad \mu_s \circ \pi^{-1}(B(\xi, r)) \leq C(\mu_t \circ \pi^{-1}(B(\xi, r)))^\alpha$$

for every  $s \in (t - \delta, t + \delta)$  and for all radii  $r > 0$  small enough. The constant  $\alpha$  is called the powering exponent of  $t$  and  $\xi$ . The following is one of the main results of our paper.

**Theorem 12.1.** *Let  $\mathcal{S}$  be a finitely primitive strongly regular conformal GDMS. Assume that both  $\mathcal{S}$  is (WBT) and parameter  $b_S$  is powering at some point  $z \in J_S$  which is either*

- (a) *not pseudo-periodic or else*
- (b) *uniquely periodic and belongs to  $\text{Int}X$  (and  $z = \pi(\xi^\infty)$  for a (unique) irreducible word  $\xi \in E_A^*$ ).*

*Then*

$$(12.2) \quad \lim_{r \rightarrow 0} \frac{\text{HD}(J_S) - \text{HD}(K_z(r))}{\mu_b(\pi^{-1}(B(z, r)))} = \begin{cases} 1/\chi_{\mu_b} & \text{if (a) holds} \\ (1 - |\varphi'_\xi(z)|)/\chi_{\mu_b} & \text{if (b) holds.} \end{cases}$$

*Proof.* Denote the right-hand side of (11.45) by  $\xi(z)$ . Put

$$h := \text{HD}(J_S) = b_S \quad \text{and} \quad h_r := \text{HD}(K_z(r)).$$

Seeking contradiction assume that (12.10) fails to hold at some point  $z \in E_A^\infty$ . This means that there exists a strictly decreasing sequence  $(s_n(z))_{n=0}^\infty$  of positive reals such that the sequence

$$\left( \frac{h - h_{s_n(z)}}{\mu_b(\pi^{-1}(B(\pi(z), s_n(z))))} \right)_{n=0}^\infty$$

does not have  $\xi(z)$  as its accumulation point. Let

$$\mathcal{R} := \{s_n(z) : n \geq 0\}.$$

Let  $(U_n^\pm(z))_{n=0}^\infty$  be the corresponding sequence of open subsets of  $E_A^\infty$  produced in formula (10.5). We shall prove the following.

**Claim 1<sup>0</sup>:** Both sequences  $(U_n^\pm(z))_{n=0}^\infty$  satisfy the (U2\*) condition for the parameter  $h$ .

*Proof.* Let  $\alpha > 0$  be a powering exponent of  $h = b_S$  at  $z$  and let  $\delta > 0$  come from this powering property. Let  $s \in (h - \delta, h + \delta)$ . Applying then formula (10.6) to the measure  $\mu_h$ , we get, with notation used in this formula, that

$$\mu_s(U_k^\pm(z)) \leq \mu_s \circ \pi^{-1}(B(z, r_{j-1})) \leq C(\mu_h \circ \pi^{-1}(B(z, r_{j-1})))^\alpha \leq C \exp^\alpha(\kappa(1+8\Delta l(z)))e^{-\alpha\kappa k}.$$

The claim is proved.  $\square$

By this claim and because of Propositions 10.8 and 10.9, Proposition 11.25 applies to give

$$(12.3) \quad \lim_{n \rightarrow \infty} \frac{h - h_n^\pm}{\mu_b(U_n^\pm(z))} = \xi(z),$$

where  $h_n^\pm := \text{HD}(K(U_n^\pm(z)))$ . Let  $(n_j)_{j=0}^\infty$  be the sequence produced in Proposition 10.7 with the help of  $\mathcal{R}$ . By virtue of this proposition there exists an increasing sequence  $(j_k)_{k=1}^\infty$  such that  $\mathcal{R} \cap \mathcal{R}_{n_{j_k}} \neq \emptyset$  for all  $k \geq 1$ . For every  $k \geq 1$  pick one element  $r_k \in \mathcal{R} \cap \mathcal{R}_{n_{j_k}}$ . Set  $q_k := l_{n_{j_k}}$ . By Observation 10.3 and formula (10.4), we have

$$(12.4) \quad \begin{aligned} \frac{h - h_{q_k}^-}{\mu_b(U_{q_k}^-(z))} \cdot \frac{\mu_b(U_{q_k}^-(z))}{\mu_b(\pi^{-1}(B(\pi(z), r_k)))} &\leq \frac{h - h_{r_k}}{\mu_b(\pi^{-1}(B(\pi(z), r_k)))} \leq \\ &\leq \frac{h - h_{q_k}^+}{\mu_b(U_{q_k}^+(z))} \cdot \frac{\mu_b(U_{q_k}^+(z))}{\mu_b(\pi^{-1}(B(\pi(z), r_k)))} \end{aligned}$$

But since  $\mu_b \circ \pi^{-1}$  is WBT, it is DBT by Proposition 9.12, and it therefore follows from (9.4) along with formulas (10.1) and (10.4) that

$$\lim_{k \rightarrow \infty} \frac{\mu_b(U_{q_k}^-(z))}{\mu_b(\pi^{-1}(B(\pi(z), r_k)))} = 1 = \lim_{k \rightarrow \infty} \frac{\mu_b(U_{q_k}^+(z))}{\mu_b(\pi^{-1}(B(\pi(z), r_k)))}.$$

Inserting this and (12.3) to (12.4) yields

$$\lim_{k \rightarrow \infty} \frac{h - h_{r_k}}{\mu_b(\pi^{-1}(B(\pi(z), r_k)))} = \xi(z).$$

Since  $r_k \in \mathcal{R}$  for all  $k \geq 1$ , this implies that  $\xi(z)$  is an accumulation point of the sequence

$$\left( \frac{h - h_{s_n(z)}}{\mu_b(\pi^{-1}(B(\pi(z), s_n(z))))} \right)_{n=0}^\infty,$$

and this contradiction finishes the proof of Theorem 12.1.  $\square$

We have discussed at length the (WBT) condition in Section 9, particularly in Theorem 9.7; we now would like also to note that since any two measures  $\mu_t$ ,  $t > \theta_S$ , are either equal or mutually singular, the standard covering argument gives the following simple but remarkable result.

**Proposition 12.2.** *If  $\mathcal{S}$  is a finitely primitive regular conformal GDMS, then every parameter  $t > \theta_S$  is powering with exponent 1 at  $\mu_t \circ \pi^{-1}$ -a.e. point of  $J_S$ .*



Now, as an immediate consequence of Theorem 12.1, Theorem 9.7, and Proposition 12.2, we get the following result, also one of our main.

**Corollary 12.3.** *If  $\mathcal{S}$  be a finitely primitive strongly regular conformal GDMS whose limit set  $J_{\mathcal{S}}$  is geometrically irreducible, then*

$$(12.5) \quad \lim_{r \rightarrow 0} \frac{\text{HD}(J_{\mathcal{S}}) - \text{HD}(K_z(r))}{\mu_b(\pi^{-1}(B(z, r)))} = \frac{1}{\chi_{\mu_b}}$$

at  $\mu_{b_{\mathcal{S}}} \circ \pi^{-1}$ -a.e. point  $z$  of  $J_{\mathcal{S}}$ .

In the case of finite alphabet  $E$ , we can say much more for the parameter  $b_{\mathcal{S}}$  than established in Proposition 12.2. Namely, we shall prove the following.

**Proposition 12.4.** *If  $\mathcal{S}$  is a finite alphabet primitive conformal GDMS, then  $\mathcal{S}$  is powering at the parameter  $b_{\mathcal{S}}$  at each point  $\xi \in J_{\mathcal{S}}$ .*

*Proof.* The proof of Theorem 7.20 in [4] (see also Theorem 7.17 therein for the main geometric ingredient of this proof) produces for every radius  $r \in (0, \frac{1}{2} \min\{\text{diam}(X_v) : v \in V\})$  a family  $Z(r) \subseteq E_A^*$  consisting of mutually incomparable words with the following properties.

- (1)  $C_1^{-1}r \leq \|\varphi'_\omega\|_\infty, \text{diam}(\varphi_\omega(X_{t(\omega)})) \leq C_1r$  for all  $\omega \in Z(r)$
- (2)  $\varphi_\omega(X_{t(\omega)}) \cap B(\xi, r) \neq \emptyset$  for all  $\omega \in Z(r)$
- (3)  $\pi_{\mathcal{S}}^{-1}(B(\xi, r)) \subseteq \bigcup_{\omega \in Z(r)} [\omega]$ ,
- (4)  $\#Z(r) \leq C_2$ ,

where  $C_1$  and  $C_2$  are some finite positive constants independent of  $\xi$  and  $r$ . Abbreviate

$$b := b_{\mathcal{S}}.$$

It easily follows from [14] that there exist a constant  $\delta \in (0, b_{\mathcal{S}}/4)$  and a constant  $Q \in (1, +\infty)$  such that

$$Q^{-1} \leq \frac{\mu_s([\tau])}{e^{-P(s)|\tau|} \|\varphi'_\tau\|_\infty^s} \leq Q$$

for every  $s \in (b - \delta, b + \delta)$  and for all  $\tau \in E_A^*$ . We therefore get for every  $s \in (b - \delta, b + \delta)$  and all  $\omega \in Z(r)$  that

$$(12.6) \quad \mu_s([\omega]) \leq Q e^{-P(s)|\omega|} \|\varphi'_\omega\|_\infty^s \leq Q C_1^s e^{-P(s)|\omega|} r^s$$

and

$$(12.7) \quad \mu_b([\omega]) \geq Q C_1^{-b} r^b.$$

It is also known from [4] that, with perhaps larger  $Q \geq 1$ :

$$(12.8) \quad \mu_b \circ \pi_{\mathcal{S}}^{-1}(B(\xi, r)) \geq Q^{-1} r^b$$

This formula follows for example from (12.7) applied to a sufficiently small fixed fraction of  $r$ . If  $b/2 \leq s \leq b$ , then  $P(s) \geq 0$ , and we get

$$\begin{aligned}
 \mu_s([\omega]) &\leq QC_1^s r^s \leq QC_1^b r^s = QC_1^b (r^b)^{s/b} \\
 &\leq QC_1^b Q^{\frac{s}{b}} \mu_b^{\frac{s}{b}} \circ \pi_S^{-1}(B(\xi, r)) \\
 (12.9) \quad &\leq Q^2 C_1^b \mu_b^{\frac{s}{b}} \circ \pi_S^{-1}(B(\xi, r)) \\
 &\leq Q^2 C_1^b \mu_b^{\frac{1}{2}} \circ \pi_S^{-1}(B(\xi, r)).
 \end{aligned}$$

Now we assume that  $s \geq b$ . We set

$$\kappa := \max\{\|\varphi'_e\|_\infty : e \in E\} < 1,$$

and we recall that

$$\chi_b := \chi_{\mu_b} = - \int_{E_A^\infty} \log |\varphi'_{\omega_0}(\pi_S(\sigma(\omega)))| d\mu_b(\omega) > 0.$$

By taking  $\delta \in (0, b/4)$  small enough, we will have

$$\frac{s - \frac{b}{2}}{s - b} \geq \frac{2\chi_b}{\log(1/\kappa)} \quad \text{and} \quad P(s) \geq -2\chi_b(s - b)$$

for all  $s \in (b, b + \delta)$ . Hence

$$\left(s - \frac{b}{2}\right) \log \kappa \leq -2\chi_b(s - b) \leq P(s).$$

Equivalently  $\kappa^{(s - \frac{b}{2})} \leq e^{P(s)}$ . Thus

$$\kappa^{(s - \frac{b}{2})|\omega|} \leq e^{P(s)|\omega|}.$$

As  $\|\varphi'_\omega\|_\infty \leq \kappa^{|\omega|}$  and  $s \geq b$ , we therefore get

$$\begin{aligned}
 \mu_s([\omega]) &\leq Qe^{-P(s)|\omega|} \|\varphi'_\omega\|_\infty^s \leq Q\|\varphi'_\omega\|_\infty^{\frac{b}{2}} \leq QC_1^{\frac{b}{2}} r^{\frac{b}{2}} \\
 &\leq Q^2 C_1^{\frac{b}{2}} Q^{\frac{b}{2}} \mu_b^{\frac{1}{2}} \circ \pi_S^{-1}(B(\xi, r)) \\
 &= Q^{3/2} C_1^{\frac{b}{2}} \mu_b^{\frac{1}{2}} \circ \pi_S^{-1}(B(\xi, r)) \\
 &\leq Q^2 C_1^b \mu_b^{\frac{1}{2}} \circ \pi_S^{-1}(B(\xi, r)).
 \end{aligned}$$

Combining this along with (12.9) we get that

$$\mu_s([\omega]) \leq Q^2 C_1^b \mu_b^{\frac{1}{2}} \circ \pi_S^{-1}(B(\xi, r)).$$

for all  $s \in (b - \delta, b + \delta)$  and all  $\omega \in Z(r)$ . Thus, looking also up at (4) and (3), this yields

$$\mu_s \circ \pi_S^{-1}(B(\xi, r)) \leq C_2 Q^2 C_1^b \mu_b^{\frac{1}{2}} \circ \pi_S^{-1}(B(\xi, r))$$

for all  $s \in (b - \delta, b + \delta)$  and all radii  $r \in (0, \frac{1}{2} \min\{\text{diam}(X_v) : v \in V\})$ . The proof of Proposition 12.4 is complete.  $\square$

As an immediate consequence of Theorem 12.1, Theorem 9.9, and Proposition 12.4, we get the following considerably stronger/fuller result.

**Theorem 12.5.** *Let  $\mathcal{S} = \{\varphi_e\}_{e \in E}$  be a primitive Conformal Graph Directed Markov System with a finite alphabet  $E$  acting in the space  $\mathbb{R}^d$ ,  $d \geq 1$ . Assume that either  $d = 1$  or that the system  $\mathcal{S}$  is geometrically irreducible. Let  $z \in J_{\mathcal{S}}$  be arbitrary. If either  $z$  is*

- (a) *not pseudo-periodic or else*
- (b) *uniquely periodic and belongs to  $\text{Int}X$  (and  $z = \pi(\xi^\infty)$  for a (unique) irreducible word  $\xi \in E_A^*$ ).*

Then

$$(12.10) \quad \lim_{r \rightarrow 0} \frac{\text{HD}(J_{\mathcal{S}}) - \text{HD}(K_z(r))}{\mu_b(\pi^{-1}(B(z, r)))} = \begin{cases} 1/\chi_{\mu_b} & \text{if (a) holds} \\ (1 - |\varphi'_\xi(z)|)/\chi_{\mu_b} & \text{if (b) holds.} \end{cases}$$

### 13. ESCAPE RATES FOR CONFORMAL PARABOLIC GDMSS

In this section, following [15] and [14], we first shall provide the appropriate setting and basic properties of conformal parabolic iterated function systems, and more generally of parabolic graph directed Markov systems. We then prove for them the appropriate theorems on escaping rates.

As in Section 7 there are given a directed multigraph  $(V, E, i, t)$  ( $E$  countable,  $V$  finite), an incidence matrix  $A : E \times E \rightarrow \{0, 1\}$ , and two functions  $i, t : E \rightarrow V$  such that  $A_{ab} = 1$  implies  $t(b) = i(a)$ . Also, we have nonempty compact metric spaces  $\{X_v\}_{v \in V}$ . Suppose further that we have a collection of conformal maps  $\varphi_e : X_{t(e)} \rightarrow X_{i(e)}$ ,  $e \in E$ , satisfying the following conditions:

- (1) (Open Set Condition)  $\varphi_i(\text{Int}(X)) \cap \varphi_j(\text{Int}(X)) = \emptyset$  for all  $i \neq j$ .
- (2)  $|\varphi'_i(x)| < 1$  everywhere except for finitely many pairs  $(i, x_i)$ ,  $i \in E$ , for which  $x_i$  is the unique fixed point of  $\varphi_i$  and  $|\varphi'_i(x_i)| = 1$ . Such pairs and indices  $i$  will be called parabolic and the set of parabolic indices will be denoted by  $\Omega$ . All other indices will be called hyperbolic. We assume that  $A_{ii} = 1$  for all  $i \in \Omega$ .
- (3)  $\forall n \geq 1 \forall \omega = (\omega_1, \dots, \omega_n) \in E^n$  if  $\omega_n$  is a hyperbolic index or  $\omega_{n-1} \neq \omega_n$ , then  $\varphi_\omega$  extends conformally to an open connected set  $W_{t(\omega_n)} \subseteq \mathbb{R}^d$  and maps  $W_{t(\omega_n)}$  into  $W_{i(\omega_n)}$ .
- (4) If  $i$  is a parabolic index, then  $\bigcap_{n \geq 0} \varphi_{i^n}(X) = \{x_i\}$  and the diameters of the sets  $\varphi_{i^n}(X)$  converge to 0.
- (5) (Bounded Distortion Property)  $\exists K \geq 1 \forall n \geq 1 \forall \omega = (\omega_1, \dots, \omega_n) \in E^n \forall x, y \in V$  if  $\omega_n$  is a hyperbolic index or  $\omega_{n-1} \neq \omega_n$ , then

$$\frac{|\varphi'_\omega(y)|}{|\varphi'_\omega(x)|} \leq K.$$

- (6)  $\exists s < 1 \forall n \geq 1 \forall \omega \in E_A^n$  if  $\omega_n$  is a hyperbolic index or  $\omega_{n-1} \neq \omega_n$ , then  $\|\varphi'_\omega\| \leq s$ .

- (7) (Cone Condition) There exist  $\alpha, l > 0$  such that for every  $x \in \partial X \subseteq \mathbb{R}^d$  there exists an open cone  $\text{Con}(x, \alpha, l) \subseteq \text{Int}(X)$  with vertex  $x$ , central angle of Lebesgue measure  $\alpha$ , and altitude  $l$ .
- (8) There exists a constant  $L \geq 1$  such that

$$||\varphi'_i(y)| - |\varphi'_i(x)|| \leq L||\varphi'_i|||y - x|$$

for every  $i \in I$  and every pair of points  $x, y \in V$ .

We call such a system of maps

$$\mathcal{S} = \{\varphi_i : i \in E\}$$

a subparabolic iterated function system. Let us note that conditions (1),(3),(5)-(7) are modeled on similar conditions which were used to examine hyperbolic conformal systems. If  $\Omega \neq \emptyset$ , we call the system  $\{\varphi_i : i \in E\}$  parabolic. As declared in (2) the elements of the set  $E \setminus \Omega$  are called hyperbolic. We extend this name to all the words appearing in (5) and (6). It follows from (3) that for every hyperbolic word  $\omega$ ,

$$\varphi_\omega(W_{t(\omega)}) \subseteq W_{t(\omega)}.$$

Note that our conditions ensure that  $\varphi'_i(x) \neq 0$  for all  $i \in E$  and all  $x \in X_{t(i)}$ . It was proved (though only for IFSs but the case of GDMSs can be treated completely similarly) in [15] (comp. [14]) that

$$(13.1) \quad \lim_{n \rightarrow \infty} \sup_{|\omega|=n} \{\text{diam}(\varphi_\omega(X_{t(\omega)}))\} = 0.$$

As its immediate consequence, we record the following.

**Corollary 13.1.** *The map  $\pi : E_A^\infty \rightarrow X := \bigoplus_{v \in V} X_v$ ,  $\{\pi(\omega)\} := \bigcap_{n \geq 0} \varphi_{\omega|_n}(X)$ , is well defined, i.e. this intersection is always a singleton, and the map  $\pi$  is uniformly continuous.*

As for hyperbolic (attracting) systems the limit set  $J = J_{\mathcal{S}}$  of the system  $\mathcal{S} = \{\varphi_e\}_{e \in E}$  is defined to be

$$J_{\mathcal{S}} := \pi(E_A^\infty)$$

and it enjoys the following self-reproducing property:

$$J = \bigcup_{e \in E} \varphi_e(J).$$

We now, following still [35] and [14], want to associate to the parabolic system  $\mathcal{S}$  a canonical hyperbolic system  $\mathcal{S}^*$ . The set of edges is this.

$$E_* := \{i^n j : n \geq 1, i \in \Omega, i \neq j \in E, A_{ij} = 1\} \cup (E \setminus \Omega) \subseteq E_A^*.$$

We set

$$V_* = t(E_*) \cup i(E_*)$$

and keep the functions  $t$  and  $i$  on  $E_*$  as the restrictions of  $t$  and  $i$  from  $E_A^*$ . The incidence matrix  $A_* : E_* \times E_* \rightarrow \{0, 1\}$  is defined in the natural (the only reasonable) way by declaring that  $A_{ab}^* = 1$  if and only if  $ab \in E_A^*$ . Finally

$$\mathcal{S}^* := \{\varphi_e : X_{t(e)} \rightarrow X_{t(e)} : e \in E_*\}.$$

It immediately follows from our assumptions (see [35] and [14] for details) that the following is true.

**Theorem 13.2.** *The system  $\mathcal{S}^*$  is a hyperbolic conformal GDMS and the limit sets  $J_{\mathcal{S}}$  and  $J_{\mathcal{S}^*}$  differ only by a countable set.*

We have the following quantitative result, whose complete proof can be found in [29].

**Proposition 13.3.** *Let  $\mathcal{S}$  be a conformal parabolic GDMS. Then there exists a constant  $C \in (0, +\infty)$  and for every  $i \in \Omega$  there exists some constant  $\beta_i \in (0, +\infty)$  such that for all  $n \geq 1$  and for all  $z \in X_i := \bigcup_{j \in I \setminus \{i\}} \varphi_j(X)$ ,*

$$C^{-1}n^{-\frac{\beta_i+1}{\beta_i}} \leq |\varphi'_{i^n}(z)| \leq Cn^{-\frac{\beta_i+1}{\beta_i}}.$$

*In fact we know more: if  $d = 2$  then all constants  $\beta_i$  are integers  $\geq 1$  and if  $d \geq 3$  then all constants  $\beta_i$  are equal to 1.*

Let

$$\beta = \beta_{\mathcal{S}} := \min\{\beta_i : i \in \Omega\}$$

Passing to equilibrium/Gibbs states and their escape rates, we now describe the class of potentials we want to deal with. This class is somewhat narrow as we restrict ourselves to geometric potentials only. There is no obvious natural larger class of potentials for which our methods would work and trying to identify such classes would be of dubious value and unclear benefits. We thus only consider potentials of the form

$$E_A^\infty \ni \omega \mapsto \zeta_t(\omega) := t \log |\varphi'_{\omega_0}(\pi_{\mathcal{S}}(\sigma(\omega)))| \in \mathbb{R}, \quad t \geq 0.$$

We then define the potential  $\zeta_t^* : E_{*A^*}^\infty \rightarrow \mathbb{R}$  as

$$\zeta_t^*(i^n j \omega) = \sum_{k=0}^n \zeta_t(\sigma^k(i^n j \omega)), \quad i \in \Omega, \quad n \geq 0, \quad j \neq i \quad \text{and} \quad i^n j \omega \in E_{*A^*}^\infty.$$

We shall prove the following.

**Proposition 13.4.** *If  $\mathcal{S}$  is a conformal parabolic GDMS, then the potential  $\zeta_t^*$  is Hölder continuous for each  $t \geq 0$  it is summable if and only if*

$$t > \frac{\beta}{\beta + 1}$$

*Proof.* Hölder continuity of potentials  $\zeta_t^*$ ,  $t \geq 0$ , follows from the fact that the system  $\mathcal{S}^*$  is hyperbolic, particularly from its distortion property, while the summability statement immediately follows from Proposition 13.3.  $\square$

So, for every  $t > \frac{\beta}{\beta+1}$  we can define  $\mu_t^*$  to be the unique equilibrium/Gibbs state for the potential  $\zeta_t^*$  with respect to the shift map  $\sigma_* : E_{*A^*}^\infty \rightarrow E_{*A^*}^\infty$ . We will not use this information in the current paper but we would like to note that  $\mu_t^*$  gives rise to a Borel  $\sigma$ -finite, unique up to multiplicative constant,  $\sigma$ -invariant measure  $\mu_t$  on  $E_A^\infty$ , absolutely continuous, in fact equivalent, with respect to  $\mu_t^*$ ; see [14] for details in the case of  $t = b_S = b_{S^*}$ , the Bowen's parameter of the systems  $\mathcal{S}$  and  $\mathcal{S}^*$  alike. The case of all other  $t > \frac{\beta}{\beta+1}$  can be treated similarly. It follows from [14] that the measure  $\mu_t$  is finite if and only if either

- (a)  $t \in \left(\frac{\beta}{\beta+1}, b_S\right)$  or
- (b)  $t = b_S$  and  $b_S > \frac{2\beta}{\beta+1}$ .

Now having all of this, as an immediate consequence of theorems Theorem 10.10 and Theorem 10.11 we get the following two results.

**Theorem 13.5.** *Let  $\mathcal{S} = \{\varphi_e\}_{e \in E}$  be a parabolic Conformal Graph Directed Markov System. Fix  $t > \frac{\beta}{\beta+1}$  and assume that the measure  $\mu_t^* \circ \pi_{S^*}^{-1}$  is (WBT) at a point  $z \in J_{S^*}$ . If  $z$  is either*

- (a) *not pseudo-periodic with respect to the system  $\mathcal{S}^*$ ,*  
or
- (b) *uniquely periodic with respect to  $\mathcal{S}^*$ , it belongs to  $\text{Int}X$  (and  $z = \pi_{S^*}(\xi^\infty)$  for a (unique) irreducible word  $\xi \in E_{*A^*}^*$ ),*

*then, with  $\underline{R}_{S^*, \mu_t^*}(B(z, \varepsilon)) := \underline{R}_{\mu_t^*}(\pi_{S^*}^{-1}(B(z, \varepsilon)))$  and  $\overline{R}_{S^*, \mu_t^*}(B(z, \varepsilon)) := \overline{R}_{\mu_t^*}(\pi_{S^*}^{-1}(B(z, \varepsilon)))$ , we have*

$$(13.2) \quad \lim_{\varepsilon \rightarrow 0} \frac{\underline{R}_{S^*, \mu_t^*}(B(z, \varepsilon))}{\mu_t^* \circ \pi_{S^*}^{-1}(B(z, \varepsilon))} = \lim_{\varepsilon \rightarrow 0} \frac{\overline{R}_{S^*, \mu_t^*}(B(z, \varepsilon))}{\mu_t^* \circ \pi_{S^*}^{-1}(B(z, \varepsilon))} = d_\varphi(z) := \begin{cases} 1 & \text{if (a) holds} \\ 1 - |\varphi'_\xi(z)|e^{-pP_{S^*}(t)} & \text{if (b) holds,} \end{cases}$$

where in (b),  $\{\xi\} = \pi_{S^*}^{-1}(z)$  and  $p \geq 1$  is the prime period of  $\xi$  under the shift map  $\sigma_* : E_{*A^*}^\infty \rightarrow E_{*A^*}^\infty$ .

**Theorem 13.6.** *Let  $\mathcal{S} = \{\varphi_e\}_{e \in E}$  be a parabolic Conformal Graph Directed Markov System whose limit set  $J_S$  is geometrically irreducible. If  $t > \frac{\beta}{\beta+1}$  then*

$$(13.3) \quad \lim_{\varepsilon \rightarrow 0} \frac{\underline{R}_{S^*, \mu_t^*}(B(z, \varepsilon))}{\mu_t^* \circ \pi_{S^*}^{-1}(B(z, \varepsilon))} = \lim_{\varepsilon \rightarrow 0} \frac{\overline{R}_{S^*, \mu_t^*}(B(z, \varepsilon))}{\mu_t^* \circ \pi_{S^*}^{-1}(B(z, \varepsilon))} = 1$$

for  $\mu_t \circ \pi^{-1}$ -a.e. point of  $J_S$ .

Sticking to notation of Section 12, given  $z \in E_{*A^*}^\infty$  and  $r > 0$  let

$$K_z^*(r) := \pi_{S^*}(\tilde{K}_z^*(r)),$$

where

$$\tilde{K}_z^*(r) := \{\omega \in E_{*A^*}^\infty : \forall_{n \geq 0} \sigma_*^n(\omega) \notin \pi_{S^*}^{-1}(B(\pi_{S^*}(z), r))\} = \bigcap_{n=0}^{\infty} \sigma_*^{-n}(\pi_{S^*}^{-1}(B(\pi_{S^*}(z), r))).$$

As immediate consequences respectively of Theorem 12.1 and Corollary 12.3, we get the following two results.

**Theorem 13.7.** *Let  $\mathcal{S} = \{\varphi_e\}_{e \in E}$  be a parabolic Conformal Graph Directed Markov System. Assume that both  $\mathcal{S}^*$  is (WBT) and parameter  $b_{\mathcal{S}}$  is powering at some point  $z \in J_{\mathcal{S}^*}$ . If  $z$  is either*

- (a) *not pseudo-periodic with respect to the system  $\mathcal{S}^*$ ,*  
or
- (b) *uniquely periodic with respect to  $\mathcal{S}^*$ , it belongs to  $\text{Int}X$  (and  $z = \pi_{\mathcal{S}^*}(\xi^\infty)$  for a (unique) irreducible word  $\xi \in E_{*A^*}^*$ ),*

then,

$$(13.4) \quad \lim_{r \rightarrow 0} \frac{\text{HD}(J_{\mathcal{S}}) - \text{HD}(K_z^*(r))}{\mu_b^*(\pi_{\mathcal{S}^*}^{-1}(B(z, r)))} = \begin{cases} 1/\chi_{\mu_b^*} & \text{if (a) holds} \\ (1 - |\varphi'_\xi(z)|)/\chi_{\mu_b^*} & \text{if (b) holds} \end{cases}.$$

**Theorem 13.8.** *Let  $\mathcal{S} = \{\varphi_e\}_{e \in E}$  be a parabolic Conformal Graph Directed Markov System whose limit set  $J_{\mathcal{S}}$  is geometrically irreducible. Then*

$$(13.5) \quad \lim_{r \rightarrow 0} \frac{\text{HD}(J_{\mathcal{S}}) - \text{HD}(K_z^*(r))}{\mu_b^*(\pi_{\mathcal{S}^*}^{-1}(B(z, r)))} = \frac{1}{\chi_{\mu_b^*}}$$

for  $\mu_{b_{\mathcal{S}}}^* \circ \pi^{-1}$ -a.e. point  $z$  of  $J_{\mathcal{S}}$ .

### Part 3. Applications: Escape Rates for Multimodal Interval Maps and One-Dimensional Complex Dynamics

Our goal in this part of the manuscript is to get the existence of escape rates in the sense of (1.1) and (1.2) for all topologically exact piecewise smooth multimodal maps of the interval  $[0, 1]$ , all rational functions of the Riemann sphere  $\hat{\mathbb{C}}$  with degree  $\geq 2$ , and a vast class of transcendental meromorphic functions from  $\mathbb{C}$  to  $\hat{\mathbb{C}}$ . In order to do this we employ two primary tools. The first one is formed by the escape rates results for the class of all countable alphabet conformal graph directed Markov systems obtained in Sections 10 and 12. The other one is the method based on the first return (induced) map developed in Section 14, Section 15, and Section 16 of this part. This method closely relates the escape rates of the original map and the induced one. It turns out that for the above mentioned class of systems one can find a set of positive measure which gives rise to the first returned map which is isomorphic to a countable alphabet conformal IFS or full shift map; the task highly non-trivial and technically involved in general. In conclusion, the existence of escape rates in the sense of (1.1) and (1.2) follows.

## 14. FIRST RETURN MAPS

Let  $(X, \rho)$  be a metric space and let  $F \subseteq X$  be a Borel set. Let  $T : X \rightarrow X$  be a Borel map. Define

$$F_\infty := F \cap \bigcap_{k=0}^{\infty} \bigcup_{n=k}^{\infty} T^{-k}(F),$$

i. e.  $F_\infty$  is the set of all those points in  $F$  that return to  $F$  infinitely often under the iteration of the map  $T$ . Then for every  $x \in F_\infty$  the number

$$\tau_F(x) := \min\{n \geq 1 : T^n(x) \in F\} = \min\{n \geq 1 : T^n(x) \in F_\infty\}$$

is well-defined, i. e. it is finite. The number  $\tau_F(x)$  is called the first return of  $x$  to  $F$  under the map  $F$ . Having the function  $\tau_F : F_\infty \rightarrow \mathbb{N}_1$  defined one defines the first return map  $T_F : F_\infty \rightarrow F_\infty$  by the formula

$$(14.1) \quad T_F(x) : T^{\tau_F(x)}(x) \in F_\infty \subseteq F.$$

Let  $B$  be a Borel subsets of  $F$ . As in the previous section let

$$K(B) = K_T(B) := \bigcap_{n=0}^{\infty} T^{-n}(X \setminus B) \quad \text{and} \quad K_F(B) := \bigcap_{n=0}^{\infty} T_F^{-n}(F_\infty \setminus B).$$

A straightforward observation is that  $K_F(B) = F_\infty \cap K(B)$ , so that we have the following.

$$(14.2) \quad K_F(B) = F_\infty \cap K(B) \subseteq F \cap K(B).$$

We shall prove the following.

**Theorem 14.1.** *If the map  $T : X \rightarrow X$  is locally bi-Lipschitz and  $B \subseteq F$  are Borel subsets of  $X$ , then*

$$\text{HD}(K(B)) = \max\{\text{HD}(K_F(B)), \text{HD}(K(F))\}.$$

*Proof.* Since  $K(F) \subseteq K(B)$ , we have that  $\text{HD}(K(F)) \leq \text{HD}(K(B))$ , and by (14.2) we have  $\text{HD}(K_F(B)) \leq \text{HD}(K(B))$ . We are thus let to show only that

$$\text{HD}(K(B)) \leq \max\{\text{HD}(K_F(B)), \text{HD}(K(F))\}.$$

Indeed, fix  $x \in K(B)$ . Let

$$N_x := \min\{n \geq 0 : T^n(x) \in F\}.$$

Consider two cases:

Case 1<sup>0</sup>: The set  $N_x$  is finite. Denote then by  $n_x$  its largest element. Then  $T^{n_x+1}(x) \in K(F)$ . Hence

$$x \in \bigcup_{n=0}^{\infty} T^{-n}(K(F));$$

note that this relation holds even if  $N_x = \emptyset$ .



Case 1<sup>0</sup>: The set  $N_x$  is infinite. Then there exists  $m_x \geq 0$  such that  $T^{m_x}(x) \in F_\infty$ . Hence,

$$x \in \bigcup_{n=0}^{\infty} T^{-n}(F_\infty).$$

In conclusion

$$K(B) \subseteq \bigcup_{n=0}^{\infty} T^{-n}(K(F)) \cup \bigcup_{n=0}^{\infty} T^{-n}(F_\infty).$$

But then, using (14.2), we get

$$\begin{aligned} K(B) &\subseteq \left( \bigcup_{n=0}^{\infty} T^{-n}(K(F)) \right) \cup \left( K(B) \cap \bigcup_{n=0}^{\infty} T^{-n}(F_\infty) \right) \\ &\subseteq \bigcup_{n=0}^{\infty} T^{-n}(K(B) \cap K(F)) \cup \bigcup_{n=0}^{\infty} T^{-n}(K(B) \cap F_\infty) \\ &= \bigcup_{n=0}^{\infty} T^{-n}(K(F)) \cup \bigcup_{n=0}^{\infty} T^{-n}(K_F(B)) \end{aligned}$$

Therefore, using  $\sigma$ -stability of Hausdorff dimension and local bi-Lipschitzness of  $T$ , we get

$$\begin{aligned} \text{HD}(K(B)) &\leq \sup_{n \geq 0} \{ \max\{ \text{HD}(T^{-n}(K(F))), \text{HD}(T^{-n}(K_F(B))) \} \} \\ &\leq \sup_{n \geq 0} \{ \max\{ \text{HD}(T^{-n}(K(F))), \text{HD}(T^{-n}(K_F(B))) \} \} \\ &= \max\{ \text{HD}(K(F)), \text{HD}(K_F(B)) \} \end{aligned}$$

The proof of Theorem 14.1 is complete.  $\square$

As an immediate consequence of this theorem we get the following.

**Corollary 14.2.** *If the map  $T : X \rightarrow X$  is locally bi-Lipschitz,  $B \subseteq F$  are Borel subsets of  $X$ , and  $\text{HD}(K(F)) < \text{HD}(K(B))$ , then*

$$\text{HD}(K(B)) = \text{HD}(K_F(B)).$$

## 15. FIRST RETURN MAPS AND ESCAPING RATES, I

As in Section 14  $(X, \rho)$  is a metric space,  $F \subseteq X$  be a Borel set and  $T : X \rightarrow X$  is a Borel map. The symbols  $F_\infty$ ,  $\tau_F$ , and  $T_F$  have the same meaning as in Section 14. Now in addition we also assume that the system  $T : X \rightarrow X$  preserves a Borel probability measure  $\mu$  on  $X$ . It is well-known that then the first return map  $T_F : F_\infty \rightarrow F_\infty$  preserves the conditional measure  $\mu_F$  on  $F$  (or  $F_\infty$  alike). This measure is given by the formula

$$\mu_F(A) = \frac{\mu(A)}{\mu(F)}$$

for every Borel set  $A \subseteq F$ . The famous Kac's Formula tells us that

$$\int_F \tau_F d\mu_F = \frac{1}{\mu(F)}.$$

For every  $n \geq 1$  denote

$$\tau_F^{(n)} := \sum_{j=0}^{n-1} \tau_F \circ T_F^j,$$

so that

$$T_F^n(x) = T_F^{(n)}(x)(x).$$

If  $B$ , as in Section 14, is a Borel subset of  $F$ , then for every  $n \geq 1$  we denote

$$B_n^c := \bigcap_{j=0}^{n-1} T^{-j}(X \setminus B), \quad B_n^c(F) := F_\infty \cap B_n^c, \quad \text{and} \quad B_n^c(T_F) := \bigcap_{j=0}^{n-1} T_F^{-j}(X \setminus B).$$

For every  $\eta \in (0, 1)$  and every integer  $k \geq 1$  denote

$$F_{k-1}(\eta) := \left\{ x \in F_\infty : \left( \frac{1}{\mu(F)} - \eta \right) k \leq \tau_F^{(k)}(x) \leq \left( \frac{1}{\mu(F)} + \eta \right) k \right\}.$$

Let us record the following straightforward observation.

$$(15.1) \quad F_{n-1}(\eta) \cap B_{\left(\frac{1}{\mu(F)} + \eta\right)n}^c \subseteq F_{n-1}(\eta) \cap B_n^c(T_F) \subseteq F_{n-1}(\eta) \cap B_{\left(\frac{1}{\mu(F)} - \eta\right)n}^c.$$

This simple relation will be however our starting point for relating the escape rates of  $B$  with respect to the map  $T$  and the first return map  $T_F : F_\infty \rightarrow F_\infty$ .

**Definition 15.1.** We say that the pair  $(T, F)$  satisfies the large deviation property (LDP) if for all  $\eta \in (0, 1)$  there exist two constants  $\hat{\eta} > 0$  and  $C_\eta \in [1, +\infty)$  such that

$$\mu(F_n^c(\eta)) \leq C_\eta e^{-\hat{\eta}n}$$

for all integers  $n \geq 1$ .

In what follows we will need one (standard) concept more. We define for every  $x \in X$  the number

$$E_F(x) := \min \{ n \in \{0, 1, 2, \dots, \infty\} : T^n(x) \in F \}.$$

This number is called the first entrance time to  $F$  under the map  $T$  and it is closely related to  $\tau_F$ ,

$$\tau_F(x) = E_F(T(x)) + 1$$

if  $x \in F$ , but of course it is different.

**Definition 15.2.** We say that the pair  $(T, F)$  has exponential tail decay (ETD) if

$$\mu(E_F^{-1}([n, +\infty])) \leq C e^{-\alpha n}$$

for all integers  $n \geq 0$  and some constants  $C, \alpha \in (0, +\infty)$ .

Let  $B$  be a Borel subset of  $F$ . Following the previous sections denote respectively by  $R_{T,\mu}(B)$  and  $R_{T_F,\mu}(B)$  the respective escape rates of  $B$  by the maps  $T : X \rightarrow X$  and  $T_F : F_\infty \rightarrow F_\infty$ , i. e.

$$\underline{R}_{T,\mu}(B) := - \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \mu(B_n^c) \leq \overline{R}_{T,\mu}(B) := - \underline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \mu(B_n^c),$$

and

$$\begin{aligned} \underline{R}_{T_F,\mu}(B) &:= - \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \mu_F(B_n^c(T_F)) = - \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \mu(B_n^c(T_F)), \\ \overline{R}_{T_F,\mu}(B) &:= - \underline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \mu_F(B_n^c(T_F)) = - \underline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \mu(B_n^c(T_F)), \end{aligned}$$

with obvious inequality

$$\underline{R}_{T_F,\mu}(B) \leq \overline{R}_{T_F,\mu}(B).$$

We shall prove the following.

**Theorem 15.3.** *Assume that a pair  $(T, F)$  satisfies the large deviation property (LDP) and has exponential tail decay (ETD). Let  $(B_k)_{k=0}^\infty$  be a sequence of Borel subsets of  $F$  such that*

- (a)  $\lim_{k \rightarrow \infty} \mu(B_k) = 0$ ,
- (b) *The limits*

$$\lim_{k \rightarrow \infty} \frac{\underline{R}_{T_F,\mu}(B_k)}{\mu_F(B_k)} \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{\overline{R}_{T_F,\mu}(B_k)}{\mu_F(B_k)}$$

*exist, are equal, and belong to  $(0, +\infty)$ ; denote their common value by  $R_F(\mu)$ .*

*Then the limits*

$$\lim_{k \rightarrow \infty} \frac{\underline{R}_{T,\mu}(B_k)}{\mu(B_k)} \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{\overline{R}_{T,\mu}(B_k)}{\mu(B_k)}$$

*also exist, and, denoting their common value by  $R_T(\mu)$ , we have that*

$$R_T(\mu) = R_F(\mu).$$

*Proof.* Fix  $\eta, \varepsilon \in (0, 1)$ . Fix two integers  $k, n \geq 1$ . Denote the sets  $(B_k)^c_{(\frac{1}{\mu(F)} + \eta)n}$  and  $(B_k)^c_{(\frac{1}{\mu(F)} - \eta)n}$  respectively by  $B_{k,n}^-(\eta)$  and  $B_{k,n}^+(\eta)$ . Because of (15.1), we have

$$(15.2) \quad \mu(F_{n-1}(\eta) \cap (B_k)^c_n(T_F)) \leq \mu(B_{k,n}^+(\eta)).$$

Fix  $M_1 \geq 1$  so large that

$$(15.3) \quad (1 - \varepsilon)R_F(\mu) \leq \frac{\underline{R}_{T_F,\mu}(B_k)}{\mu_F(B_k)} \leq \frac{\overline{R}_{T_F,\mu}(B_k)}{\mu_F(B_k)} \leq (1 + \varepsilon)R_F(\mu)$$

and

$$(15.4) \quad 4R_F(\mu)\mu_F(B_k) \leq \min\{\varepsilon, \hat{\eta}/2\}$$

for all  $k \geq M_1$ . Fix such a  $k$ . Fix then  $N_k \geq 1$  so large that

$$\exp\left(-(1+\varepsilon)\overline{R}_{T_F,\mu}(B_k)n\right) \leq \mu((B_k)_n^c(T_F)) \leq \exp\left(-(1-\varepsilon)\underline{R}_{T_F,\mu}(B_k)n\right)$$

for all  $n \geq N_k^{(1)}$ . Along with (15.3) this gives

$$(15.5) \quad \exp\left(-(1+e)^2 R_F(\mu)\mu_F(B_k)n\right) \leq \mu((B_k)_n^c(T_F)) \leq \exp\left(-(1-\varepsilon)^2 R_F(\mu)\mu_F(B_k)n\right).$$

Therefore, using also Definition 15.1, we get for all  $k \geq M_1$  and all  $n \geq N_k^{(1)}$  that

$$\begin{aligned} \frac{\mu(F_{n-1}(\eta) \cap (B_k)_n^c(T_F))}{\mu((B_k)_n^c(T_F))} &= \frac{\mu((B_k)_n^c(T_F)) - \mu(F_{n-1}^c(\eta) \cap (B_k)_n^c(T_F))}{\mu((B_k)_n^c(T_F))} \\ &\geq \frac{\mu((B_k)_n^c(T_F)) - \mu(F_{n-1}^c(\eta))}{\mu((B_k)_n^c(T_F))} = 1 - \frac{\mu(F_{n-1}^c(\eta))}{\mu((B_k)_n^c(T_F))} \\ &\geq 1 - \frac{C_\eta e^{-\hat{\eta}(n-1)}}{\exp\left(-4R_F(\mu)\mu_F(B_k)n\right)} \\ &= 1 - C_\eta e^{\hat{\eta}} \exp\left((4R_F(\mu)\mu_F(B_k) - \hat{\eta})n\right) \\ &\geq 1 - C_\eta e^{\hat{\eta}} \exp\left(-\frac{1}{2}\hat{\eta}n\right) \geq 1/2, \end{aligned}$$

where the last inequality holds for all  $n \geq N_k^{(1)}$  large enough, say  $n \geq N_k^{(2)} \geq N_k^{(1)}$ . Along with (15.2) this gives

$$\mu(B_{k,n}^+(\eta)) \geq \frac{1}{2}\mu((B_k)_n^c(T_F)).$$

Hence

$$-\lim_{n \rightarrow \infty} \frac{1}{n} \log \mu(B_{k,n}^+(\eta)) \leq \overline{R}_{T_F,\mu}(B_k).$$

Since

$$(B_k)_n^c \supseteq B_{k, \left\lfloor \frac{n}{\mu(F) - \eta} \right\rfloor + 1}^+,$$

we get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mu(B_{k,n}^+(\eta)) \leq \left(\frac{1}{\mu(F)} - \eta\right) \lim_{n \rightarrow \infty} \frac{1}{n} \log \mu((B_k)_n^c).$$

Therefore,

$$\overline{R}_{T_F,\mu}(B_k) \geq \left(\frac{1}{\mu(F)} - \eta\right) \overline{R}_{T,\mu}(B_k).$$

Dividing both sides of this inequality by  $\mu(B_k)$  and passing to the limit with  $k \rightarrow \infty$ , this entails

$$R_F(\mu) \geq (1 - \eta\mu(F)) \overline{\lim}_{k \rightarrow \infty} \frac{\overline{R}_{T,\mu}(B_k)}{\mu(B_k)}.$$

By letting in turn  $\eta \searrow 0$ , this yields

$$(15.6) \quad R_F(\mu) \geq \varlimsup_{k \rightarrow \infty} \frac{\overline{R}_{T,\mu}(B_k)}{\mu(B_k)}.$$

Passing to the proof of the opposite inequality, denote  $\left(\frac{1}{\mu(F)} + \eta\right)n$  by  $n^+$  and  $\left(\frac{1}{\mu(F)} + \eta\right)^{-1}n$  by  $n^-$ . We have

$$(15.7) \quad \begin{aligned} B_{k,n}^-(\eta) &= \bigcup_{j=0}^{n^+} (B_{k,n}^-(\eta) \cap E_F^{-1}(j)) \cup E_F^{-1}((n^+, +\infty]) \\ &= E_F^{-1}((n^+, +\infty]) \cup \bigcup_{j=0}^{n^+} E_F^{-1}(j) \cap T^{-j}((B_k)_{n^+-j}^c). \end{aligned}$$

Now,

$$(15.8) \quad \begin{aligned} E_F^{-1}(j) \cap T^{-j}((B_k)_{n^+-j}^c) &= E_F^{-1}(j) \cap T^{-j}(F) \cap T^{-j}((B_k)_{n^+-j}^c) \\ &= E_F^{-1}(j) \cap T^{-j}(F \cap (B_k)_{n^+-j}^c) \\ &= E_F^{-1}(j) \cap \left( T^{-j}(F_{(n^+-j)^- - 1}(\eta) \cap (B_k)_{n^+-j}^c) \cup T^{-j}(F_{(n^+-j)^- - 1}^c(\eta) \cap (B_k)_{n^+-j}^c) \right). \end{aligned}$$

By (15.1) we have

$$(15.9) \quad \begin{aligned} F_{(n^+-j)^- - 1}^c(\eta) \cap (B_k)_{n^+-j}^c &= F_{(n^+-j)^- - 1}^c(\eta) \cap (B_k)_{((n^+-j)^-)^+}^c \\ &\subseteq F_{(n^+-j)^- - 1}^c(\eta) \cap (B_k)_{(n^+-j)^-}^c(T_F) \\ &\subseteq (B_k)_{(n^+-j)^-}^c(T_F). \end{aligned}$$

Now take  $p, q > 1$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ . By applying Hölder inequality,  $T$ -invariantness of the measure  $\mu$ , (15.5), and making use of Definition 15.2, we get for all  $0 \leq j \leq N_k^{(1)}$ , that

$$\begin{aligned}
\mu\left(E_F^{-1}(j) \cap T^{-j}(F_{(n^+-j)^- - 1}(\eta) \cap (B_k)_{n^+-j}^c)\right) &\leq \mu\left(E_F^{-1}(j) \cap T^{-j}((B_k)_{(n^+-j)^-}^c(T_F))\right) = \\
&= \int_X \mathbb{1}_{E_F^{-1}(j)} \mathbb{1}_{T^{-j}((B_k)_{(n^+-j)^-}^c(T_F))} d\mu \\
&\leq \left(\int_X \mathbb{1}_{E_F^{-1}(j)} d\mu\right)^{1/p} \left(\int_X \mathbb{1}_{T^{-j}((B_k)_{(n^+-j)^-}^c(T_F))} d\mu\right)^{1/q} \\
&= \mu^{1/p}(E_F^{-1}(j)) \mu^{1/q}(T^{-j}((B_k)_{(n^+-j)^-}^c(T_F))) \\
&= \mu^{1/p}(E_F^{-1}(j)) \mu^{1/q}((B_k)_{(n^+-j)^-}^c(T_F)) \\
&\leq C^{1/p} e^{-\frac{\alpha}{p}j} \exp\left(-\frac{(1-\varepsilon)^2}{q} R_F(\mu) \mu_F(B_k) (n^+ - j)^-\right) \\
&= C^{1/p} e^{-\frac{\alpha}{p}j} \exp\left(-\frac{(1-\varepsilon)^2}{q} R_F(\mu) \mu_F(B_k) \left(\frac{1}{\mu(F)} + \eta\right)^{-1} \left(\left(\frac{1}{\mu(F)} + \eta\right) n - j\right)\right) \\
&= C^{1/p} \exp\left(-\frac{(1-\varepsilon)^2}{q} R_F(\mu) \mu_F(B_k) n\right) \\
&\quad \cdot \exp\left(-\left(\frac{\alpha}{p} - \frac{(1-\varepsilon)^2}{q} \left(\frac{1}{\mu(F)} + \eta\right)^{-1} R_F(\mu) \mu_F(B_k)\right) j\right)
\end{aligned}$$

Together with the left-hand side of (15.5) this gives that

$$\begin{aligned}
\frac{\mu\left(E_F^{-1}(j) \cap T^{-j}(F_{(n^+-j)^- - 1}(\eta) \cap (B_k)_{n^+-j}^c)\right)}{\mu((B_k)_n^c(T_F))} &\leq \\
&\leq C^{1/p} \exp\left(R_F(\mu) \mu_F(B_k) \left((1+e)^2 - \frac{1}{q}(1-\varepsilon)^2\right) n\right) \\
&\quad \cdot \exp\left(-\left(\frac{\alpha}{p} - \frac{(1-\varepsilon)^2}{q} \left(\frac{1}{\mu(F)} + \eta\right)^{-1} R_F(\mu) \mu_F(B_k)\right) j\right).
\end{aligned}$$

Taking now  $q > 1$  sufficiently close to 1 and looking at (15.4) we will have for every  $\varepsilon > 0$  small enough that

$$(1+e)^2 - \frac{1}{q}(1-\varepsilon)^2 \leq 1 - \frac{1}{q} + 6\varepsilon \leq 7\varepsilon \quad \text{and} \quad \frac{\alpha}{p} - \frac{(1-\varepsilon)^2}{q} \left(\frac{1}{\mu(F)} + \eta\right)^{-1} R_F(\mu) \mu_F(B_k) > \frac{\alpha}{2p}.$$

Therefore, for all  $k \geq M_1$  and for all  $0 \leq j \leq n^+ - N_k^{(1)}$ , we have that

$$(15.10) \quad \frac{\mu\left(E_F^{-1}(j) \cap T^{-j}(F_{(n^+-j)^- - 1}(\eta) \cap (B_k)_{n^+-j}^c)\right)}{\mu((B_k)_n^c(T_F))} \leq C^{1/p} e^{7\varepsilon n} e^{-\frac{\alpha}{2p}j}.$$

For  $n^+ - N_k^{(1)} < j \leq n^+$ , using (15.4), the left-hand side of (15.5), and looking up at Definition 15.2, we have the easier estimate:

$$\begin{aligned}
 (15.11) \quad & \frac{\mu\left(E_F^{-1}(j) \cap T^{-j}\left(F_{(n^+-j)^- - 1}(\eta) \cap (B_k)_{n^+-j}^c\right)\right)}{\mu\left((B_k)_n^c(T_F)\right)} \leq \frac{\mu\left(E_F^{-1}(j)\right)}{\mu\left((B_k)_n^c(T_F)\right)} \leq \\
 & \leq C^{-\alpha j} \exp\left((1+e)^2 R_F(\mu) \mu_F(B_k) n\right) \\
 & \leq C \exp\left(-\alpha(n^+ - N_k^{(1)})\right) \exp\left((1+e)^2 R_F(\mu) \mu_F(B_k) n\right) \\
 & = C e^{\alpha N_k^{(1)}} \exp\left(\left((1+e)^2 R_F(\mu) \mu_F(B_k) - \alpha\left(\frac{1}{\mu(F)} + \eta\right)\right) n\right) \\
 & \leq C e^{\alpha N_k^{(1)}}.
 \end{aligned}$$

Now we can estimate the second part of (15.8). We note that

$$E_F^{-1}(j) \cap T^{-j}\left(F_{(n^+-j)^- - 1}(\eta) \cap (B_k)_{n^+-j}^c\right) \subseteq E_F^{-1}(j) \cap T^{-j}\left(F_{(n^+-j)^- - 1}(\eta)\right),$$

and use again Hölder inequality,  $T$ -invariance of measure  $\mu$ , and Definitions 15.2 and 15.1, to estimate:

$$\begin{aligned}
 & \mu\left(E_F^{-1}(j) \cap T^{-j}\left(F_{(n^+-j)^- - 1}(\eta) \cap (B_k)_{n^+-j}^c\right)\right) \leq \\
 & \leq \mu\left(E_F^{-1}(j) \cap T^{-j}\left(F_{(n^+-j)^- - 1}(\eta)\right)\right) = \int_X \mathbb{1}_{E_F^{-1}(j)} \mathbb{1}_{T^{-j}\left(F_{(n^+-j)^- - 1}(\eta)\right)} d\mu \\
 & \leq \mu^{1/p}(E_F^{-1}(j)) \cdot \nu^{1/q}\left(T^{-j}\left(F_{(n^+-j)^- - 1}(\eta)\right)\right) \\
 & = \mu^{1/p}(E_F^{-1}(j)) \cdot \nu^{1/q}\left(F_{(n^+-j)^- - 1}(\eta)\right) \\
 & \leq C^{1/p} e^{-\frac{\alpha}{p}j} C_\eta^{1/q} e^{-\frac{\hat{\eta}}{q}\left((n^+-j)^- - 1\right)} \\
 & = C^{1/p} C_\eta^{1/q} e^{\frac{\hat{\eta}}{q}e^{-\frac{\alpha}{p}j}} \exp\left(-\left(\frac{\alpha}{p} - \frac{\hat{\eta}}{q}\left(\frac{1}{\mu(F)} + \eta\right)^{-1}\right)j\right).
 \end{aligned}$$

Combining this with the left-hand side of (15.5) this gives that

$$\begin{aligned}
 (15.12) \quad & \frac{\mu\left(E_F^{-1}(j) \cap T^{-j}\left(F_{(n^+-j)^- - 1}(\eta) \cap (B_k)_{n^+-j}^c\right)\right)}{\mu\left((B_k)_n^c(T_F)\right)} \\
 & \leq C^{1/p} C_\eta^{1/q} e^{\frac{\hat{\eta}}{q}} \exp\left(-\left(\frac{\hat{\eta}}{q} - (1+e)^2 R_F(\mu) \mu_F(B_k)\right)n\right) \cdot \\
 & \cdot \exp\left(-\left(\frac{\alpha}{p} - \frac{\hat{\eta}}{q}\left(\frac{1}{\mu(F)} + \eta\right)^{-1}\right)j\right).
 \end{aligned}$$

Now, first take  $q > 1$  so large that

$$\frac{\alpha}{p} - \frac{\hat{\eta}}{q}\left(\frac{1}{\mu(F)} + \eta\right)^{-1} > \frac{\alpha}{2}.$$

Then take  $k \geq M_1$ , say  $k \geq M_{1,q} \geq M_1$  so large that

$$\frac{\hat{\eta}}{q} - (1+e)^2 R_F(\mu) \mu_F(B_k) \geq \frac{\hat{\eta}}{2q}.$$

Inserting these two inequalities into (15.12), yields

$$(15.13) \quad \frac{\mu\left(E_F^{-1}(j) \cap T^{-j}\left(F_{(n^+-j)^- - 1}(\eta) \cap (B_k)_n^c\right)\right)}{\mu\left((B_k)_n^c(T_F)\right)} \leq C^{1/p} C_\eta^{1/q} e^{\frac{\hat{\eta}}{q}} e^{-\frac{\hat{\eta}}{2q}n} e^{-\frac{\alpha}{2}j}.$$

Finally, by Definition 15.2, the left-hand side of (15.5), and (15.4),

$$\begin{aligned} \frac{m\left(E_F^{-1}((n^+, +\infty])\right)}{\mu\left((B_k)_n^c(T_F)\right)} &\leq C e^{-\alpha n^+} \exp\left((1+e)^2 R_F(\mu) \mu_F(B_k) n\right) \\ &= C \exp\left(-\left(\alpha \left(\frac{1}{\mu(F)} + \eta\right) - (1+e)^2 R_F(\mu) \mu_F(B_k) n\right)\right) \\ &\leq C \end{aligned}$$

for every  $\varepsilon > 0$  small enough and  $n \geq M_1$ . Combining this inequality, (15.10), (15.11), (15.9), (15.8), and (15.7), we get for every  $k \geq 1$  large enough, every  $e > 0$ , and every  $n \geq N_k^{(1)}$ , that

$$\frac{\mu(B_{k,n}^-(\eta))}{\mu\left((B_k)_n^c(T_F)\right)} \leq C(1 + e^{\alpha N_k^{(1)}}) + C' \sum_{j=0}^{n^+} e^{-\frac{\alpha}{2}j} + C'' e^{7\varepsilon n} \sum_{j=0}^{n^+ - N_k^{(1)}} e^{-\frac{\alpha}{2p}j} \leq C''' e^{7\varepsilon n}$$

with some constants  $C, C', C'', C''' \in (0, +\infty)$  and  $p > 1$  independent of  $k \geq 1$  large enough,  $n \geq N_k^{(1)}$ , and  $\varepsilon \in (0, 1)$  small enough. Hence,

$$- \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \mu(B_{k,n}^-(\eta)) \geq \underline{R}_{T_F, \mu}(B_k) - 7\varepsilon$$

for every  $\varepsilon > 0$  and every  $k \geq 1$  large enough. Therefore,

$$- \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \mu(B_{k,n}^-(\eta)) \geq \underline{R}_{T_F, \mu}(B_k)$$

for every  $k \geq 1$  large enough. Since

$$(B_k)_n^c \subseteq B_k^- \left[ \frac{n}{\frac{1}{\mu(F)} + \eta} \right],$$

we get

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \mu((B_k)_n^c) \leq \frac{1}{\frac{1}{\mu(F)} + \eta} \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \mu(B_{k,n}^-(\eta)).$$

Therefore,

$$\underline{R}_{T_F, \mu}(B_k) \leq \left( \frac{1}{\mu(F)} + \eta \right) \underline{R}_{T, \mu}(B_k)$$



for every  $k \geq 1$  large enough. Dividing both sides of this inequality by  $\mu(B_k)$  and passing to the limit with  $k \rightarrow \infty$ , this gives

$$R_F(\mu) \leq (1 + \eta\mu(F)) \lim_{k \rightarrow \infty} \frac{R_{T,\mu}(B_k)}{\mu(B_k)}.$$

By letting in turn  $\eta \searrow 0$ , this yields

$$R_F(\mu) \leq \lim_{k \rightarrow \infty} \frac{R_{T,\mu}(B_k)}{\mu(B_k)}.$$

Together with (15.6) this finishes the proof of Theorem 15.3.  $\square$

## 16. FIRST RETURN MAPS AND ESCAPING RATES, II

In this section we keep the settings of Section 15; more specifically that described between its beginning until formula (15.1). In particular, we do not assume appriori that (LDP) holds. In fact our goal in this section is provide natural sufficient conditions for (LDP) to hold. Let  $\varphi : X \rightarrow \mathbb{R}$  be a Borel measurable function. We define the function  $\varphi_F : F \rightarrow \mathbb{R}$  by the formula

$$(16.1) \quad \varphi_F(x) := \sum_{j=0}^{\tau_F(x)-1} \varphi \circ T^j(x).$$

It is well-known

$$(16.2) \quad \int_X \varphi d\mu = \mu(F) \int_F \varphi_F d\mu_F.$$

In particular,

$$\mathbb{1}_F = \tau_F,$$

and, inserting this to (16.2), we obtain the familiar, discussed in the previous section, Kac's Formula

$$\int_F \tau_F d\mu_F = \frac{1}{\mu(F)}.$$

**Definition 16.1.** We say that a pentadde  $(X, T, F, \mu, \varphi)$ , or just  $T$ , is of symbol return type (SRT) if the following conditions are satisfied:

- (a)  $F = E_A^\infty$  for some countable alphabet  $E$  and some finitely irreducible incidence matrix  $A$ .
- (b)  $T_F = \sigma : E_A^\infty \rightarrow E_A^\infty$ .
- (c)  $\varphi_F : F \rightarrow \mathbb{R}$  is a Hölder continuous summable potential.
- (d)  $P(\varphi_F) = 0$ .
- (e)  $\mu = \mu_{\varphi_F}$  is the Gibbs/equilibrium state for the potential  $\varphi_F : F \rightarrow \mathbb{R}$

(f) There are two constants  $C, \alpha > 0$  such that

$$\mu(\tau_F^{-1}(n)) \leq Ce^{-\alpha n}$$

for all integers  $n \geq 1$ .

Since

$$\tau_F^{-1}(n) \subseteq T^{-1}(E_F^{-1}(n-1))$$

and since the measure  $\mu$  is  $T$ -invariant, we immediately obtain the following.

**Observation 16.2.** If a pentadde  $(X, T, F, \mu, \varphi)$  satisfies all conditions (a)–(e) of Definition 16.1 and it also has exponential tail decay (ETD), then  $(X, T, F, \mu, \varphi)$  also satisfies condition (f) of Definition 16.1; thus in conclusion, the pentadde  $(X, T, F, \mu, \varphi)$  is then of symbol return type (SRT).

Given  $\theta \in \mathbb{R}$  we consider the potential

$$\varphi_\theta := \varphi_F + \theta\tau_F : F \rightarrow \mathbb{R}.$$

We shall prove several lemmas. We start with the following.

**Lemma 16.3.** *If  $T$  is an (SRT) system, then the potential  $\varphi_\theta : F \rightarrow \mathbb{R}$  is summable for every  $\theta < \alpha$ .*

*Proof.* Since  $T$  is SRT, we have that

$$\begin{aligned} \sum_{e \in E} \exp(\sup(\varphi_\theta|_{[e]})) &= \sum_{e \in E} \exp(\sup((\varphi_F + \theta\tau_F)|_{[e]})) \\ &= \sum_{e \in E} \exp(\sup(\varphi_F|_{[e]})) \exp(\theta\tau_F(e)) \asymp \sum_{e \in E} \mu([e]) \exp(\theta\tau_F(e)) \\ &= \sum_{n=1}^{\infty} \sum_{\tau_F(e)=n} \mu([e]) e^{\theta n} = \sum_{n=1}^{\infty} e^{\theta n} \sum_{\tau_F(e)=n} \mu([e]) \\ &= \sum_{n=1}^{\infty} e^{\theta n} \mu(\tau_F^{-1}(n)) \leq C \sum_{n=1}^{\infty} \exp((\theta - \alpha)n) < +\infty, \end{aligned}$$

whenever  $\theta < \alpha$ . The proof is complete.  $\square$

**Lemma 16.4.** *If  $T$  is an (SRT) system, then the function  $(-\infty, \alpha) \ni \theta \mapsto P(\varphi_\theta) \in \mathbb{R}$  is real-analytic.*

*Proof.* In the terminology of Corollary 2.6.10 in [14], condition (c) of Definition 16.1 says that  $\varphi_F \in \mathcal{K}_\beta$ , where  $\beta > 0$  is the Hölder exponent of  $\varphi_F$ . Of course  $\tau_F \in \mathcal{K}_\beta$  since  $\tau_F$  is constant on cylinders of length one. Lemma 16.3 says that  $\varphi_\theta \in \mathcal{K}_\beta$  for all  $\theta < \alpha$ ; in fact the proof of this lemma shows that  $\varphi_\theta \in \mathcal{K}_\beta$  for all  $\theta \in \mathbb{C}$  with  $\operatorname{Re}(\theta) < \alpha$ . This now means that all hypotheses of Corollary 2.6.10 from [14] are satisfied. The upshot of this corollary is that the function

$$\{\theta \in \mathbb{C} : \operatorname{Re}(\theta) < \alpha\} \ni \theta \mapsto \mathcal{L}_{\varphi_\theta} \in L(\mathcal{K}_\beta)$$

is holomorphic, where  $\mathcal{L}_{\varphi_\theta}$  is the Perron-Frobenius operator associated to the potential  $\varphi_\theta$  and the shift map  $\sigma = T_F$ . The proof is now concluded by applying Kato-Rellich perturbation Theorem and the fact that  $\exp(P(\varphi_\theta))$  is a simple isolated eigenvalue of  $\mathcal{L}_{\varphi_\theta}$  for all real  $\theta < \alpha$  (it is not really relevant here but in fact  $\exp(P(\varphi_\theta))$  is equal to the spectral radius of the operator  $\mathcal{L}_{\varphi_\theta} \in L(\mathcal{K}_\beta)$ ), see the paragraph of [14] located between Remark 2.6.11 and Theorem 2.6.12 for more details.  $\square$

Because of Lemma 16.3, for every  $\theta < \alpha$  there exists a unique Gibbs/equilibrium state  $\mu_\theta$  for the potential  $\varphi_\theta : F \rightarrow \mathbb{R}$ . Having the previous two lemmas, Proposition 2.6.13 in [14] applies to give the following.

**Lemma 16.5.** *If  $T$  is an (SRT) system, then*

$$\frac{d}{d\theta} P(\varphi_\theta) = \int_F \tau_F d\mu_\theta$$

for every  $\theta < \alpha$ .

Now having all the three previous lemmas along with Definition 16.1, employing the standard (by now) tools of [14], exactly the same proof as in [5] yields the following.

**Theorem 16.6.** *If  $T$  is an (SRT) system, then for every  $\theta < \alpha$  we have that*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mu(\{x \in F : \text{sgn}(\theta) \tau_F^n(x) \geq \text{sgn}(\theta) n \int_F \tau_F d\mu_\theta\}) = -\theta \int_F \tau_F d\mu_\theta + P(\varphi_\theta).$$

In order to make use of this theorem we shall prove the following.

**Lemma 16.7.** *If  $T$  is an (SRT) system and the first return map function  $\tau_F : F_\infty \rightarrow \mathbb{N}$  is unbounded, then for every non-zero  $\theta < \alpha$  we have that*

$$P(\varphi_\theta) - \theta \int_F \tau_F d\mu_\theta < 0.$$

*Proof.* Since  $\mu_\theta$  is an equilibrium state for  $\varphi_\theta$ , we have that

$$\begin{aligned} P(\varphi_\theta) - \theta \int_F \tau_F d\mu_\theta &= h_{\mu_\theta}(\sigma) + \int_F \varphi_F d\mu_\theta + \theta \int_F \tau_F d\mu_\theta - \theta \int_F \tau_F d\mu_\theta \\ &= h_{\mu_\theta}(\sigma) + \int_F \varphi_F d\mu_\theta \\ &\leq P(\varphi_F) = 0. \end{aligned}$$

Hence, in order to complete the proof we only need to show that the inequality sign above is strict. In order to do this suppose for a contradiction that  $h_{\mu_\theta}(\sigma) + \int_F \varphi_F d\mu_\theta = P(\varphi_F)$ . But then the fact that  $\mu$  is the only equilibrium state for  $\varphi_F$ , implies that  $\mu_\theta = \mu$ . But because of Theorem 2.2.7 in [14] this in turn implies that the function  $\varphi_\theta - \varphi_F$  is cohomologous to a constant in the class of Hölder continuous functions defined on  $E_A^\infty = F$ . But  $\varphi_\theta - \varphi_F = \theta \tau_F$  is, by our hypotheses, unbounded unless  $\theta = 0$ . This finishes the proof.  $\square$

Now we can prove the main result of this section:

**Lemma 16.8.** *If  $T$  is an (SRT) system and the first return map function  $\tau_F : F_\infty \rightarrow \mathbb{N}$  is unbounded, then the pair  $(T_F, F)$  satisfies the large deviation property (LDP).*

*Proof.* Fix  $\eta \in (0, 1)$ . It follows from Lemma 16.4 and Lemma 16.5 that the function

$$(-\infty, \alpha) \ni \theta \mapsto \int_F \tau_F d\mu_\theta \in [1, +\infty)$$

is continuous. Therefore, there exists  $\delta \in (0, \alpha)$  such that

$$\int_F \tau_F d\mu - \eta \leq \int_F \tau_F d\mu_\delta, \int_F \tau_F d\mu_{-\delta} \leq \int_F \tau_F d\mu + \eta.$$

Equivalently:

$$\mu(F)^{-1} - \eta \leq \int_F \tau_F d\mu_\delta, \int_F \tau_F d\mu_{-\delta} \leq \mu(F)^{-1} + \eta.$$

Hence for every  $k \geq 1$ :

$$F_{k-1}^c(\eta) \subseteq \{x \in F : \tau_F^k(x) \geq k \int_F \tau_F d\mu_\delta\} \cup \{x \in F : \tau_F^k(x) \leq k \int_F \tau_F d\mu_{-\delta}\}.$$

So, denoting

$$\hat{\eta} := \frac{1}{2} \min \left\{ \delta \int_F \tau_F d\mu_\delta - P(\varphi_d), -\delta \int_F \tau_F d\mu_{-\delta} - P(\varphi_{-\delta}) \right\},$$

which is positive by Lemma 16.7, we conclude from Theorem 16.6, that

$$\mu(F_{k-1}^c(\eta)) \leq C_\eta e^{-\hat{\eta}k}$$

for all  $k \geq 1$ . The proof is complete.  $\square$

## 17. ESCAPE RATES FOR INTERVAL MAPS

In this and the next sections we will reap the benefits of our work in the previous sections, most notably of that on escape rates of conformal countable alphabet IFSs and of that on the first return map techniques including large deviations. This section is devoted to the study of the multimodal smooth maps of an interval.

We start with the definition of the class of dynamical systems and potentials we consider.

**Definition 17.1.** Let  $I = [0, 1]$  be the closed interval. Let  $T : I \rightarrow I$  be a  $C^3$  differentiable map with the following properties:

- (a)  $T$  has only a finitely many maximal closed intervals of monotonicity; or equivalently  $\text{Crit}(T) = \{x \in I : T'(x) = 0\}$ , the set of all critical points of  $T$  is finite.
- (b) The dynamical system  $T : I \rightarrow I$  is topologically exact, meaning that for every non-empty subset  $U$  of  $I$  there exists an integer  $n \geq 0$  such that  $T^n(U) = I$ .
- (c) All critical points are non-flat.

(d)  $T : I \rightarrow I$  is a *topological Collet-Eckmann map*, meaning that

$$\inf\{(|(T^n)'(x)|)^{1/n} : T^n(x) = x \text{ for } n \geq 1\} > 1$$

where the infimum is taken over all integers  $n \geq 1$  and all fixed points of  $T^n$ .

We then call  $T : I \rightarrow I$  a *topologically exact topological Collet-Eckmann map* (teTCE). If (c) and (d) are relaxed and only (a) and (b) are assumed then  $T$  is called a topologically exact multimodal map.

As in the case of rational functions we set

$$\text{PC}(T) := \bigcup_{n=1}^{\infty} T^n(\text{Crit}(T))$$

and call this the *postcritical* set of  $T$ . Again as in the case of rational functions we say that the map  $T : I \rightarrow I$  is *tame* if

$$\overline{\text{PC}(T)} \neq I.$$

The following theorem is due to many authors and a detailed and readable discussion on this topic can be found, for example, in [21]

**Theorem 17.2** (Exponential Shrinking Property). *If  $T : I \rightarrow I$  satisfies conditions (a)–(c) of Definition 17.1, then  $T$  is a (te)TCE, i.e. condition (d) holds if and only if there exist  $\delta > 0$ ,  $\gamma > 0$  and  $C > 0$  such that if  $z \in I$  and  $n \geq 0$  then*

$$\text{diam}(W) \leq Ce^{-\gamma n}$$

*for each connected component  $W$  of  $T^{-n}(B(z, 2\delta))$ .*

The hard part of this theorem is its “if” part. The converse is easy. There are more conditions equivalent to teTCE, but we need only the above Exponential Shrinking Property (ESP) and we do not bring them up here. We now however articulate two standard sufficient conditions for (ESP) to hold. It is implied by the Collet-Eckmann condition which requires that there exist  $\lambda > 1$  and  $C > 0$  such that for every integer  $n \geq 0$  we have that

$$|(f^n)'(f(c))| \geq C\lambda^n.$$

If also suffices to assume that the map  $T$  is semi-hyperbolic, i.e., that no critical points  $c$  in the Julia belongs to its own omega limit set  $\omega(c)$  for (ESP) to hold. This so for example, if  $T$  is a classical unimodal map of the form  $I \ni x \mapsto \lambda x(1 - x)$ , with  $0 < \lambda \leq 4$  and the critical point  $1/2$  is not in its own omega limit set, i.e.,  $1/2 \notin \omega(1/2)$ .

We call a potential  $\psi : I \rightarrow \mathbb{R}$  *acceptable* if it is Lipschitz continuous and

$$\sup(\psi) - \inf(\psi) < h_{\text{top}}(T).$$

We would also like to mention that for the purposes of this section it would suffice that  $\psi : I \rightarrow \mathbb{R}$  is Hölder continuous (with any exponent) and of bounded variation. We denote by  $\text{BV}_I$  the vector space of all functions in  $L^1(\lambda)$ , where  $\lambda$  denotes Lebesgue measure on  $I$ ,

that have a version of bounded variation. This vector space becomes a Banach space when endowed with the norm

$$\|g\|_{BV} := \|g\|_{\text{Leb}_1} + v_I(g)$$

where  $v_I(g)$  denotes the variation of  $g$  on  $I$ . For every  $g \in BV_I$  define the *Perron-Frobenius operator* associated to  $\psi$  by

$$\mathcal{L}_\psi(g)(x) = \sum_{y \in T^{-1}(x)} g(y) e^{\psi(y)}.$$

It is well known and easy to check that  $\mathcal{L}_\psi(BV_I) \subset BV_I$  and  $\mathcal{L}_\psi : BV_I \rightarrow BV_I$  is a bounded linear operator.

The following theorem collects together some fundamental results of [9] and [10]

**Theorem 17.3.** *If  $T : I \rightarrow I$  is a topologically exact multimodal map and  $\psi : I \rightarrow \mathbb{R}$  is an acceptable potential then*

- (a) *there exists a Borel probability eigenmeasure  $m_\psi$  for the dual operator  $\mathcal{L}_\psi^*$  whose corresponding eigenvalue is equal to  $e^{P(\psi)}$ . It then follows that  $\text{supp}(m_\psi) = I$ .*
- (b) *there exists a unique Borel  $T$ -invariant probability measure  $\mu_\psi$  on  $I$  absolutely continuous with respect to  $m_\psi$ . Furthermore,  $\mu_\psi$  is equivalent to  $m_\psi$ ;*
- (c)  *$h_{\mu_\psi}(T) + \int_I \psi d\mu_\psi = P(\psi)$ , meaning that  $\mu_\psi$  is an (ergodic) equilibrium state for  $\psi : I \rightarrow \mathbb{R}$  with respect to the dynamical system  $T : I \rightarrow I$ .*
- (d) *The Perron-Frobenius  $\mathcal{L}_\psi : BV_I \rightarrow BV_I$  is quasi-compact.*
- (e)  *$r(\mathcal{L}_\psi) = e^{P(\psi)}$ .*
- (f)  *$\text{sp}(\mathcal{L}_\psi) \cap \partial B(0, e^{P(\psi)}) = \{e^{P(\psi)}\}$*
- (g) *The number  $e^{P(\psi)}$  is a simple isolated eigenvalue (this follows from (f), (e) and (f)) of  $\mathcal{L}_\psi : BV_I \rightarrow BV_I$  with eigenfunction  $\rho_\psi := \frac{d\mu_\psi}{dm_\psi}$  which is Lipschitz continuous and log-bounded.*

We shall use the commonly accepted convention, used throughout this article, that for every  $r \in (0, 1]$  and every bounded interval  $\Delta \subset \mathbb{R}$  we denote by  $r\Delta$  the (smaller) interval of length  $r|\Delta|$  centred at the same point as  $\Delta$ . We now consider the following version of the bounded distortion property taken from [21] whose proof has a long history and is well documented therein.

**Theorem 17.4.** *Let  $T : I \rightarrow I$  be a teTCE. Then for every  $r \in (0, 1)$  there exists  $K(r) \in (0, +\infty)$  such that if  $\Delta \subset I$  is an interval,  $n \geq 0$  is an integer, the map  $T^n|_\Delta$  is 1-to-1, and  $x, y \in \Delta$  are such that  $T^n(x), T^n(y) \in rT^n(\Delta)$ , then*

$$\left| \frac{(T^n)'(y)}{(T^n)'(x)} - 1 \right| \leq K(r) |(T^n)(y) - (T^n)(x)|.$$

We next recall the following definition.

**Definition 17.5.** An interval  $V \subset I$  is called a *nice set* for a multimodal map  $T : I \rightarrow I$  if

$$\text{int}(V) \cap \bigcup_{n=0}^{\infty} T^n(\partial V) = \emptyset$$

The proof of the following theorem is both standard and straightforward, and has been presented in various similar settings. We provide the proof below because of the critical importance for us of the theorem it proves and the brevity of the proof, for the sake of completeness, and for the convenience of the reader.

**Theorem 17.6.** *If  $T : I \rightarrow I$  is topologically exact multimodal map then for every point  $\xi \in (0, 1)$  and every  $R > 0$  there exists a nice set  $V \subset I$  such that  $\xi \in V \subset B(\xi, R)$ .*

*Proof.* Since the map  $T : I \rightarrow I$  is topologically exact it has a dense set of periodic points. Fix one periodic point  $\omega$ , say of prime period  $p \geq 1$ , such that  $\xi \notin \bigcup_{k=0}^{\infty} T^{-k}(\{T^j(\omega) : 0 \leq j \leq p-1\})$ . Again because of topological exactness of  $T$ ,

$$\overline{\xi \in (0, \xi) \cap \bigcup_{k=0}^{\infty} T^{-k}(\{T^j(\omega) : 0 \leq j \leq p-1\})}$$

and

$$\overline{\xi \in (\xi, 1) \cap \bigcup_{k=0}^{\infty} T^{-k}(\{T^j(\omega) : 0 \leq j \leq p-1\})}.$$

For every  $n \geq 1$ , sufficiently large denote by  $\xi_n^- \in I$  the point closest to  $\xi$  in

$$\overline{(0, \xi) \cap \bigcup_{k=0}^n T^{-k}(\{T^j(\omega) : 0 \leq j \leq p-1\})}$$

and by  $\xi_n^+ \in I$  the point closest to  $\xi$  in

$$\overline{(\xi, 1) \cap \bigcup_{k=0}^n T^{-k}(\{T^j(\omega) : 0 \leq j \leq p-1\})}.$$

We then denote

$$V_n := (\xi_n^-, \xi_n^+).$$

Then obviously  $\xi \in V_n$ ,  $T^k(\xi_n^\pm) \notin (\xi_n^-, \xi_n^+)$  for all  $k = 0, 1, \dots, n-1$ , and  $T^k(\xi_n^\pm) \in \{T^j(\omega) : 0 \leq j \leq n-1\}$  for all  $k \geq n$ . Since  $\lim_{n \rightarrow +\infty} \xi_n^\pm = \xi$  it then follows that  $T^k(\xi_n^\pm) \notin V_n$  for all  $k \geq n$ . In conclusion,  $V_n$  are the required nice sets for all integers  $n \geq 1$ . Since in addition  $\lim_{n \rightarrow +\infty} \text{diam}(V_n) = 0$  the proof is complete.  $\square$

From their definitions, nice sets enjoy the following property.

**Theorem 17.7.** *If  $V$  is a nice set for a multimodal map, then for every integer  $n \geq 0$  and every  $U \in \text{Comp}(T^{-n}(V))$  either*

$$U \cap V = \emptyset \text{ or } U \subset V.$$

From now on throughout this section we assume that  $T : I \rightarrow I$  is a tame teTCE map. Fix a point  $\xi \in I \setminus \overline{PC(T)}$ . By virtue of Theorem 17.4 there is a nice set  $V$  such that

$$\xi \in V \text{ and } 2V \cap \overline{PC(T)} = \emptyset.$$

The nice set  $V$  canonically gives rise to a countable alphabet conformal iterated function system in the sense considered in the previous sections of the present paper. Namely, put

$$\text{Comp}_*(V) = \bigcup_{n=1}^{\infty} \text{Comp}(f^{-n}(V)).$$

For every  $U \in \text{Comp}_*(V)$  let  $\tau_V(U) \geq 1$  the unique integer  $n \geq 1$  such that  $U \in \text{Comp}(f^{-n}(V))$ . Put further

$$\varphi_U := f_U^{-\tau_V(U)} : V \rightarrow U$$

and keep in mind that

$$\varphi_U(V) = U.$$

Denote by  $E_V$  the subset of all elements  $U$  of  $\text{Comp}_*(V)$  such that

- (a)  $\varphi_U(V) \subset V$ ,
- (b)  $f^k(U) \cap V = \emptyset$  for all  $k = 1, 2, \dots, \tau_V(U) - 1$ .

The collection

$$\mathcal{S}_V := \{\varphi_U : V \rightarrow V\}$$

of all such inverse branches forms obviously an iterated function system in the sense considered in the previous sections of the present paper. In other words the elements of  $\mathcal{S}_V$  are formed by all inverse branches of the first return map  $f_V : V \rightarrow V$ . In particular,  $\tau_V(U)$  is the first return time of all points in  $U = \varphi_U(V)$  to  $V$ . We define the function  $N_V : E_V^\infty \rightarrow \mathbb{N}_1$  by setting

$$N_V(\omega) := \tau_V(\omega_1).$$

Let

$$\pi_V : E_V^\infty \rightarrow \mathbb{R}$$

be the canonical projection induced by the iterated function system  $\mathcal{S}_V$ . Let

$$J_V := \pi_V(E_V^\infty)$$

be the limit set of the system  $\mathcal{S}_V$ . Clearly

$$J_V \subseteq I.$$

It is immediate from our definitions that

$$\tau_V(\pi(\omega)) = N_V(\omega)$$

for all  $\omega \in E_V^\mathbb{N}$ .

We shall now prove the following.

**Proposition 17.8.** *Let  $T : I \rightarrow I$  be a tame teTCE map. Let  $\psi : I \rightarrow \mathbb{R}$  be an acceptable potential. Then*

- (a)  $\tilde{\psi}_V := \psi_V \circ \pi_V - P(\psi)N_V : E^\mathbb{N} \rightarrow \mathbb{R}$  is a summable Hölder continuous potential;
- (b)  $P(\sigma, \tilde{\psi}_V) = 0$  for the pressure for the shift map  $\sigma : E_V^\mathbb{N} \rightarrow E_V^\mathbb{N}$ ;
- (c)  $\mu_{\varphi,V} = \mu_{\tilde{\psi}_V} \circ \pi_V^{-1}$ , where  $\mu_{\tilde{\psi}_V}$  is the equilibrium state for  $\tilde{\psi}_V$  and the shift map  $\sigma : E_V^\mathbb{N} \rightarrow E_V^\mathbb{N}$ ;



- (d) In addition,  $\psi_V$  is the amalgamated function of a summable Hölder continuous system of functions.

*Proof.* Hölder continuity of  $\tilde{\psi}_V$  follows directly from Theorem 17.2 (the Exponential Shrinking Property) and the fact that the function  $N_V$  is constant on cylinders of length 1. Hölder continuity of  $\tilde{\psi}_V$  follows directly from Theorem 17.2 (the Exponential Shrinking Property) and the fact that the function  $N_V$  is constant on cylinders of length 1. We define a Hölder continuous system of functions  $G = \{g^{(l)} : V \rightarrow \mathbb{R}\}_{e \in E}$  by putting

$$g^{(e)} := (\psi_V - P(\varphi)\tau_V) \circ \varphi_e, \quad e \in E.$$

Theorem 17.3 then implies the system  $G$  is summable,  $P(G) = 0$ , and  $m_{\psi,V}$  is the unique  $G$ -conformal measure for the IFS  $\mathcal{S}_V$ . According to [14],  $g : E_V^{\mathbb{N}} \rightarrow \mathbb{R}$ , the amalgamated function of  $G$  is defined by the formula

$$\begin{aligned} g(\omega) &= g^{(\omega_1)}(\pi_V(\sigma(\omega))) = \psi_V \circ \varphi_{\omega_1}(\pi_V(\sigma(\omega))) - P(\psi)\tau_V \circ \varphi_{\omega_1}(\pi_V(\sigma(\omega))) \\ &= \psi_V \circ \pi_V(\omega) - P(\psi)N_V(\omega) \\ &= \tilde{\psi}_V(\omega). \end{aligned}$$

By Proposition 3.1.4 in [14] we thus have that

$$P(\sigma, \tilde{\psi}_V) = P(G) = 0.$$

Now, since  $\pi_V \circ \sigma = T_V \circ \pi_V$ , i.e. since the dynamical system  $T_V : J_V \rightarrow J_V$  is a factor of the shift map  $\sigma : E_V^{\mathbb{N}} \rightarrow E_V^{\mathbb{N}}$  via the map  $\pi_V : E_V^{\mathbb{N}} \rightarrow J_V$ , we see that  $\mu_{\tilde{\psi}_V} \circ \pi_V^{-1}$  is a Borel  $f_V$ -invariant probability measure on  $J_V$  equivalent to  $m_{\tilde{\psi}_V} \circ \pi_V^{-1} = m_g \circ \pi_V^{-1} = m_G = m_{\psi,V}$ . Since  $m_{\psi,V}$  is equivalent to  $\mu_{\psi,V}$ , we thus conclude that the measures  $m_{\tilde{\psi}_V} \circ \pi_V^{-1}$  and  $\mu_{\psi,V}$  are equivalent. Since both these measures are  $T_V$ -invariant and  $\mu_{\psi,V}$  is ergodic, they must be equal. The proof is thus complete.  $\square$

Since  $\pi_V : E_V^{\mathbb{N}} \rightarrow J_V = V_{\infty}$ , where, we recall the latter is the set of points returning infinitely often to  $V$ , is a measurable isomorphism sending the  $\sigma$ -invariant measure  $\mu_{\tilde{\psi}_V}$  to the  $f_V$ -invariant probability measure  $\mu_{\psi,V}$ , by identifying the sets  $E_V^{\mathbb{N}}$  and  $V_{\infty}(= J_V)$ , we can prove the following.

**Lemma 17.9.** *With all the hypotheses of Proposition 17.8, the pentade  $(I, T, V, \tilde{\psi}_V, \mu_{\tilde{\psi}_V})$  is an SRT system having exponential tail decay (ETD), where we recall that  $V_{\infty}$  is identified with  $E_V^{\mathbb{N}}$ ,  $\tilde{\psi}_V$  is identified with  $\psi_V - P(\psi)\tau_V$ , and  $\mu_{\tilde{\psi}_V}$  is identified with  $\mu_{\psi,V}$ .*

*Proof.* By virtue of Proposition 17.8 and Observation 16.2 we only need to prove that the pentade  $(I, T, V, \tilde{\psi}_V, \mu_{\tilde{\psi}_V})$  has exponential tail decay (ETD). We can assume without loss of generality that  $\psi : I \rightarrow \mathbb{R}$  is normalized so that

$$P(\psi) = 0 \quad \text{and} \quad m_{\psi} = \mu_{\psi}.$$

For every  $n \geq 1$  denote by  $\mathcal{C}_V(n)$  all the connected components of  $T^{-n}(V)$ . Then define

$$\mathcal{C}_V^0(n) := \{U \in \mathcal{C}_V(n) : \forall_{(0 \leq k \leq n-1)} T^k(U) \cap V = \emptyset\}$$

and

$$\mathcal{C}_V^*(n) := \{U \in \mathcal{C}_V(n) : U \subseteq V\} = \{U \in \mathcal{C}_V(n) : U \cap V \neq \emptyset\}.$$

Since the map  $T : I \rightarrow I$  is topologically exact, there exists an integer  $q \geq 1$  such that

$$T^q(V) \supseteq I.$$

Therefore for every  $e \in \mathcal{C}_V(n)$  there exists (at least one)  $\hat{e} \in \mathcal{C}_V^*(n+q)$  such that

$$T^q \circ \varphi_{\hat{e}} = \varphi_e.$$

By conformality of the measure  $\mu_\psi$ , for every  $e \in \mathcal{C}_V(n)$ , we have

$$\mu_\psi(\varphi_{\hat{e}}(V)) \geq \exp(-q\|\psi\|_\infty) \mu_\psi(\varphi_e(V)).$$

So, since

$$\bigcup_{a \in \mathcal{C}_V^0(n+q)} \varphi_a(V) \subseteq \bigcup_{\substack{b \in \mathcal{C}_V(n+q) \\ T^q \circ \varphi_b \in \mathcal{C}_V^0(n)}} \varphi_b(V) \setminus \bigcup_{e \in \mathcal{C}_V^0(n)} \varphi_{\hat{e}}(V),$$

we therefore get

$$\begin{aligned} \mu_\psi \left( \bigcup_{a \in \mathcal{C}_V^0(n+q)} \varphi_a(V) \right) &\leq \mu_\psi \left( \bigcup_{\substack{b \in \mathcal{C}_V(n+q) \\ T^q \circ \varphi_b \in \mathcal{C}_V^0(n)}} \varphi_b(V) \setminus \bigcup_{e \in \mathcal{C}_V^0(n)} \varphi_{\hat{e}}(V) \right) \\ &= \mu_\psi \left( \bigcup_{\substack{b \in \mathcal{C}_V(n+q) \\ T^q \circ \varphi_b \in \mathcal{C}_V^0(n)}} \varphi_b(V) \right) - \mu \left( \bigcup_{e \in \mathcal{C}_V^0(n)} \varphi_{\hat{e}}(V) \right) \\ &= \mu_\psi \left( T^{-q} \left( \bigcup_{c \in \mathcal{C}_V^0(n)} \varphi_c(V) \right) \right) - \sum_{e \in \mathcal{C}_V^0(n)} \mu_\psi(\varphi_{\hat{e}}(V)) \\ &= \mu_\psi \left( \bigcup_{c \in \mathcal{C}_V^0(n)} \varphi_c(V) \right) - \sum_{e \in \mathcal{C}_V^0(n)} \mu_\psi(\varphi_{\hat{e}}(V)) \\ &\leq \mu_\psi \left( \bigcup_{c \in \mathcal{C}_V^0(n)} \varphi_c(V) \right) - \exp(-q\|\psi\|_\infty) \sum_{e \in \mathcal{C}_V^0(n)} \mu_\psi(\varphi_e(V)) \\ &= \gamma \mu_\psi \left( \bigcup_{c \in \mathcal{C}_V^0(n)} \varphi_c(V) \right), \end{aligned}$$

where  $\gamma := 1 - \exp(-q\|\psi\|_\infty) \in [0, 1)$ . An immediate induction then yields

$$\mu_\psi \left( \bigcup_{e \in \mathcal{C}_V^0(n)} \varphi_e(V) \right) \leq \gamma^{-1} \gamma^{n/q}$$

for all  $n \geq 0$ . But, as

$$E_V^{-1}([n, +\infty]) = E_V^{-1}(\{+\infty\}) \cup \bigcup_{k=n}^{\infty} \bigcup_{e \in \mathcal{C}_V^0(k)} \varphi_e(V)$$

and since  $\mu_\psi(E_V^{-1}(\{+\infty\})) = 0$  by ergodicity of  $\mu_\psi$  and of  $\mu_\psi(V) > 0$ , we therefore get that

$$(17.1) \quad \mu_\psi(E_V^{-1}([n, +\infty])) \leq (\gamma(1 - \gamma^{1/q}))^{-1} \gamma^{n/q}$$

for all  $n \geq 0$ . This just means that the pentade  $(I, T, V, \tilde{\psi}_V, \mu_{\tilde{\psi}_V})$  has exponential tail decay (ETD), and the proof is complete.  $\square$

Denote by  $I_R(T)$  the set of all recurrent points of  $T$  in  $I$ . Formally

$$I_R(T) := \{z \in I : \varliminf_{n \rightarrow \infty} |T^n(z) - z| = 0\}.$$

Of course  $I_R(T) \subseteq J_T$  and  $\mu_\psi(I \setminus I_R(T)) = 0$  because of Poincaré's Recurrence Theorem. The set  $I_R(T)$  is significant for us since

$$I_R(T) \cap V \subseteq J_V.$$

Now we can harvest the fruits of the work we have done. As a direct consequence of Theorem 10.10, Theorem 10.11, Proposition 17.8, Lemma 17.9, Lemma 16.8, and Theorem 15.3, we get the following two results.

**Theorem 17.10.** *Let  $T : I \rightarrow I$  be a tame teTCE map. Let  $\psi : I \rightarrow \mathbb{R}$  be an acceptable potential. Let  $z \in I_R(T) \setminus \overline{\text{PC}(T)}$ .*

*Assume that the equilibrium state  $\mu_\psi$  is (WBT) at  $z$ . Then*

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{R_{\mu_\psi}(B(z, \varepsilon))}{\mu_\psi(B(z, \varepsilon))} &= \lim_{\varepsilon \rightarrow 0} \frac{\overline{R}_{\mu_\psi}(B(z, \varepsilon))}{\mu_\psi(B(z, \varepsilon))} = \\ &= \begin{cases} & \text{if } z \text{ is not any periodic point of } T, \\ 1 - \exp(S_p \psi(z) - pP(f, \psi)) & \text{if } z \text{ is a periodic point of } T. \end{cases} \end{aligned}$$

**Theorem 17.11.** *Let  $T : I \rightarrow I$  be a tame teTCE map. Let  $\psi : I \rightarrow \mathbb{R}$  be an acceptable potential. Then*

$$\lim_{\varepsilon \rightarrow 0} \frac{R_{\mu_\psi}(B(z, \varepsilon))}{\mu_\psi(B(z, \varepsilon))} = \lim_{\varepsilon \rightarrow 0} \frac{\overline{R}_{\mu_\psi}(B(z, \varepsilon))}{\mu_\psi(B(z, \varepsilon))} = 1$$

*for  $\mu_\psi$ -a.e. point  $z \in I$ .*

**Definition 17.12.** A multimodal map  $T : I \rightarrow I$  is called *subexpanding* if

$$\text{Crit}(T) \cap \overline{\text{PC}(T)} = \emptyset.$$

It is not hard to see (good references for a proof can be found in [21]) that the following is true.

**Proposition 17.13.** *Any topologically exact multimodal subexpanding map of the interval  $I$  is a tame teTCE map.*

Let us quote another well-known result which can be found, for example, in the book of de Melo and van Strien [18].

**Theorem 17.14.** *If  $T : I \rightarrow I$  is a topologically exact multimodal subexpanding map, then there exists a unique Borel probability  $T$ -invariant measure  $\mu$  absolutely continuous with respect to Lebesgue measure  $\lambda$ . In fact,*

- (a)  $\mu$  is equivalent to  $\lambda$  and (therefore)
- (b) has full topological support.
- (c) The Radon–Nikodym derivative  $\frac{d\mu}{d\lambda}$  is uniformly bounded above and separated from zero on the complement of every fixed neighborhood of  $\overline{\text{PC}(T)}$ .
- (d)  $\mu$  is ergodic, even  $K$ -mixing,
- (e)  $\mu$  has Rokhlin’s natural extension metrically isomorphic to some two sided Bernoulli shift and
- (f)  $\mu$  charges with full measure both topologically transitive and radial points of  $T$ .

As an immediate consequence of this theorem, particularly of its item (c), we get the following.

**Corollary 17.15.** *If  $T : I \rightarrow I$  is a topologically exact multimodal subexpanding map, then the  $T$ -invariant measure  $\mu$  absolutely continuous with respect to Lebesgue measure  $\lambda$  is (WBT) at every point of  $I \setminus \overline{\text{PC}(T)}$ .*

Passing to escape rates, by a small obvious modification (see [21] for details) of the proof of Theorem 17.6 for all  $c \in \text{Crit}(T) \cup \{\xi\}$  there are arbitrarily small open intervals  $V_c$ ,  $c \in V_c$ , such that  $V_c \cap \overline{\text{PC}(T)} = \emptyset$  and the collection  $T_*^{-n}$ ,  $n \geq 1$ , of all continuous (equivalently smooth inverse branches of  $T^n$ ) defined on  $V_c$ ,  $c \in \text{Crit}(T) \cup \{\xi\}$ , and such that for some  $c' \in \text{Crit}(T) \cup \{\xi\}$ ,

$$T_*^{-n}(V_c) = V_{c'}$$

and

$$\bigcup_{k=1}^{n-1} T^k(T_*^{-n}(V_c)) \cap \bigcup \{V_z : z \in \text{Crit}(T) \cup \{\xi\}\} = \emptyset$$

forms a finitely primitive conformal GDS, which we will call  $\mathcal{S}_T$ , whose limit set contains  $\text{Trans}(T)$ . Another characterization of  $\mathcal{S}_T$  is that its elements are composed of continuous inverse branches of the first return map of  $f$  from

$$V := \bigcup \{V_z : z \in \text{Crit}(T) \cup \{\xi\}\}$$

to  $V$ . It has been proved in [21] that  $\text{HD}(K(V)) < 1$ .

So, since by Theorem 12.1,  $\lim_{r \rightarrow 0} \text{HD}(K(B(\xi, r))) = h$ , we conclude that

$$\text{HD}(K(V)) < \text{HD}(K(B(\xi, r)))$$

for all  $r > 0$  small enough. Therefore, since  $b_{\mathcal{S}_T} = 1$  and since  $\mu_{h,V} = \mu_{b_{\mathcal{S}_f}}$ , applying Theorem 12.1, Corollary 12.3 and Corollary 14.2, we get the following two theorems.

**Theorem 17.16.** *Let  $T : I \rightarrow I$  be a topologically exact multimodal subexpanding map. Fix  $\xi \in I \setminus \overline{\text{PC}(T)}$ . Assume that the parameter 1 is powering at  $\xi$  with respect to the conformal GDS  $\mathcal{S}_T$ . Then the following limit exists, is finite, and positive:*

$$\lim_{r \rightarrow 0} \frac{1 - \text{HD}(K_\xi(r))}{\mu(B(\xi, r))}.$$

**Theorem 17.17.** *If  $T : I \rightarrow I$  is a topologically exact multimodal subexpanding map, then for Lebesgue-a.e. point  $\xi \in I \setminus \overline{\text{PC}(T)}$  the following limit exists, is finite and positive:*

$$\lim_{r \rightarrow 0} \frac{1 - \text{HD}(K_\xi(r))}{\mu(B(\xi, r))}.$$

## 18. ESCAPE RATES FOR RATIONAL FUNCTIONS OF THE RIEMANN SPHERE

Now, we will apply the results of sections 14 and 15 to two large classes of conformal dynamical systems in the complex plane: rational functions of the Riemann sphere  $\widehat{\mathbb{C}}$  in this section and, in the next section, transcendental meromorphic functions on  $\mathbb{C}$ . This section considerably overlaps in some of its parts with the previous section on the multimodal interval maps. We provide here its full exposition for the sake of coherent completeness and convenience of the readers not necessarily interested in interval maps.

As said, now we deal with rational functions. Let  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  be a rational function of degree  $d \geq 2$ . Let  $J(f)$  denote the Julia sets of  $f$  and let

$$\text{Crit}(f) := \{c \in \widehat{\mathbb{C}} : f'(c) = 0\}$$

be the set of all critical (branching) points of  $f$ . Put

$$\text{PC}(f) := \bigcup_{n=1}^{\infty} f^n(\text{Crit}(f))$$

and call it the postcritical set of  $f$ . The best understood and the easiest (nowadays) to deal with class of rational functions is formed by expanding (also frequently called hyperbolic) maps. The rational map  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  is said to be expanding if the restriction  $f|_{J(f)} : J(f) \rightarrow J(f)$  satisfies

$$(18.1) \quad \inf\{|f'(z)| : z \in J(f)\} > 1$$

or, equivalently,

$$(18.2) \quad |f'(z)| > 1$$

for all  $z \in J(f)$ . Another, topological, characterization of expandingness is this.

**Fact 18.1.** A rational function  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  is expanding if and only if

$$J(f) \cap \overline{\text{PC}(f)} = \emptyset.$$

It is immediate from this characterization that all the polynomials  $z \mapsto z^d$ ,  $d \geq 2$ , are expanding along with their small perturbations  $z \mapsto z^d + \varepsilon$ ; in fact expanding rational functions are commonly believed to form a vast majority amongst all rational functions. This is known at least for polynomials with real coefficients. We however do not restrict ourselves to expanding rational maps only. We start with all rational functions, no restriction whatsoever, and then make some, weaker than hyperbolicity, appropriate assumptions.

Let  $\psi : \widehat{\mathbb{C}} \rightarrow \mathbb{R}$  be a Hölder continuous function, referred to in the sequel as potential. We say that  $\psi : \widehat{\mathbb{C}} \rightarrow \mathbb{R}$  has a pressure gap if

$$nP(\psi) - \sup(\psi_n) > 0$$

for some integer  $n \geq 1$ , where  $P(\psi)$  denotes the ordinary topological pressure of  $\psi|_{J(f)}$  and the Birkhoff's sum  $\psi_n$  is also considered as restricted to  $J(f)$ .

We would like to mention that (1.9) always holds (with all  $n \geq 0$  sufficiently large) if the function  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  restricted to its Julia set is expanding (also frequently referred to as hyperbolic).

The probability invariant measure we are interested in comes from the following.

**Theorem 18.2** ([6]). *If  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  is a rational function of degree  $d \geq 2$  and if  $\psi : \widehat{\mathbb{C}} \rightarrow \mathbb{R}$  is a Hölder continuous potential with a pressure gap, then  $\psi$  admits a unique equilibrium state  $\mu_\psi$ , i.e. a unique Borel probability  $f$ -invariant measure on  $J(f)$  such that*

$$P(\psi) = h_{\mu_\psi}(f) + \int_{J(f)} \psi d\mu_\psi.$$

In addition,

- (a) *the measure  $\mu_\psi$  is ergodic, in fact  $K$ -mixing, and (see [29]) enjoys further finer stochastic properties.*
- (b) *The Jacobian*

$$J(f) \ni z \mapsto \frac{d\mu_\psi \circ T}{d\mu_\psi}(z) \in (0, +\infty)$$

*is a Hölder continuous function.*

In [22] a rational function  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  was called tame if

$$J(f) \setminus \overline{\text{PC}(f)} \neq \emptyset.$$

Likewise, following [25], we adopt the same definition for (transcendental) meromorphic functions  $f : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$ .

**Remark 18.3.** Tameness is a very mild hypothesis and there are many classes of maps for which these hold. These include:

- (1) Quadratic maps  $z \mapsto z^2 + c$  for which the Julia set is not contained in the real line;

- (2) Rational maps for which the restriction to the Julia set is expansive which includes the case of expanding rational functions; and
- (3) Misiurewicz maps, where the critical point is not recurrent.

In this paper the main advantage of dealing with tame functions is that these admit Nice Sets. Let us define and discuss them now.

Given a set  $F \subseteq \widehat{\mathbb{C}}$  and  $n \geq 0$ , we denote by  $\text{Comp}(f^{-n}(F))$  the collection of all connected components of  $f^{-n}(F)$ . J. Rivera-Letelier introduced in [24] the concept of Nice Sets in the realm of the dynamics of rational maps of the Riemann sphere. In [7] N. Dobbs proved their existence for tame meromorphic functions from  $\mathbb{C}$  to  $\widehat{\mathbb{C}}$ . We quote now his theorem.

**Theorem 18.4.** *Let  $f : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$  be a tame meromorphic function. Fix a non-periodic point  $z \in J(f) \setminus \overline{\text{PC}(f)}$ ,  $\kappa > 1$ , and  $K > 1$ . Then for all  $L > 1$  and for all  $r > 0$  sufficiently small there exists an open connected set  $V = V(z, r) \subseteq \mathbb{C} \setminus \overline{\text{PC}(f)}$  such that*

- (a) *If  $U \in \text{Comp}(f^{-n}(V))$  and  $U \cap V \neq \emptyset$ , then  $U \subseteq V$ .*
- (b) *If  $U \in \text{Comp}(f^{-n}(V))$  and  $U \cap V \neq \emptyset$ , then, for all  $w, w' \in U$ ,*

$$|(f^n)'(w)| \geq L \quad \text{and} \quad \frac{|(f^n)'(w)|}{|(f^n)'(w')|} \leq K.$$

- (c)  $\overline{B(z, r)} \subset U \subset B(z, \kappa r) \subseteq \mathbb{C} \setminus \overline{\text{PC}(f)}$ .

Each nice set canonically gives rise to a countable alphabet conformal iterated function system in the sense considered in the previous sections of the present paper. Namely, put

$$\text{Comp}_*(V) = \bigcup_{n=1}^{\infty} \text{Comp}(f^{-n}(V)).$$

For every  $U \in \text{Comp}_*(V)$  let  $\tau_V(U) \geq 1$  the unique integer  $n \geq 1$  such that  $U \in \text{Comp}(f^{-n}(V))$ . Put further

$$\varphi_U := f_U^{-\tau_V(U)} : V \rightarrow U$$

and keep in mind that

$$\varphi_U(V) = U.$$

Denote by  $E_V$  the subset of all elements  $U$  of  $\text{Comp}_*(V)$  such that

- (a)  $\varphi_U(V) \subseteq V$ ,
- (b)  $f^k(U) \cap V = \emptyset$  for all  $k = 1, 2, \dots, \tau_V(U) - 1$ .

The collection

$$\mathcal{S}_V := \{\varphi_U : V \rightarrow V\}$$

of all such inverse branches forms obviously a conformal iterated function system in the sense considered in the previous sections of the present paper. In other words the elements of  $\mathcal{S}_V$  are formed by all holomorphic inverse branches of the first return map  $f_V : V \rightarrow V$ .

In particular,  $\tau_V(U)$  is the first return time of all points in  $U = \varphi_U(V)$  to  $V$ . We define the function  $N_V : E_V^{\mathbb{N}} \rightarrow \mathbb{N}_1$  by setting

$$N_V(\omega) := \tau_V(\omega_1).$$

Let

$$\pi_V : E_V^{\mathbb{N}} \rightarrow \widehat{\mathbb{C}}$$

be the canonical projection induced by the iterated function system  $\mathcal{S}_V$ . Let

$$J_V : \pi_V(E_V^{\mathbb{N}})$$

be the limit set of the system  $\mathcal{S}_V$ . Clearly

$$J_V \subseteq J(f).$$

It is immediate from our definitions that

$$\tau_V(\pi(\omega)) = N_V(\omega)$$

for all  $\omega \in E_V^{\mathbb{N}}$ .

Now, having in addition a Hölder continuous potential  $\psi : \widehat{\mathbb{C}} \rightarrow \mathbb{R}$  with pressure gap, we already know from the previous sections that  $\mu_{\psi,V}$ , the conditional measure of  $\mu_{\psi}$  on  $V$  is  $f_V$ -invariant and ergodic.

**Definition 18.5.** We say that the rational function  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  has the Exponential Shrinking Property (ESP) if there exist  $\delta > 0$ ,  $\gamma > 0$ , and  $C > 0$  such that if  $z \in J(f)$  and  $n \geq 0$ , then

$$(18.3) \quad \text{diam}(W) \leq Ce^{-\gamma n}$$

for each  $W \in \text{Comp}(f^{-n}(B(z, 2\delta)))$ .

**Remark 18.6.** This property has been thoroughly explored in the papers including [20] and the references therein. These papers provide several different characterizations of Exponential Shrinking Property, most notably the one called Topological Collet-Eckmann; one of them being uniform hyperbolicity of periodic points in the Julia set. We do not recall any more of them here as we will only need (ESP).

We now however articulate two standard sufficient conditions for (ESP) to hold. It is implied by the Collet-Eckmann condition which requires that there exist  $\lambda > 1$  and  $C > 0$  such that for every integer  $n \geq 0$  we have that

$$|(f^n)'(f(c))| \geq C\lambda^n.$$

If also suffices for (ESP) to hold to assume that a rational map is semi-hyperbolic, i.e., that no critical point  $c$  in the Julia belongs to its own omega limit set  $\omega(c)$ . This so for example, if  $T$  is a classical unimodal map of the form  $I \ni x \mapsto \lambda x(1-x)$ , with  $0 < \lambda \leq 4$  and the critical point  $1/2$  is not in its own omega limit set, i.e.,  $1/2 \notin \omega(1/2)$ .

Last observation: all expanding rational functions have the Exponential Shrinking Property (ESP).

We shall prove the following.



**Proposition 18.7.** *Let  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  be a tame rational function satisfying (ESP). Let  $\psi : \widehat{\mathbb{C}} \rightarrow \mathbb{R}$  be a Hölder continuous potential with pressure gap. If  $V$  is a nice set for  $f$ , then*

(a)

$$\tilde{\psi}_V := \psi_V \circ \pi_V - P(\psi)N_V : E_V^{\mathbb{N}} \rightarrow \mathbb{R}$$

*is a Hölder continuous potential,*

(b)  $P(\sigma, \tilde{\psi}_V) = 0$ ,

(c)

$$\mu_{\psi,V} = \mu_{\tilde{\psi}_V} \circ \pi_V^{-1},$$

*where  $\mu_{\tilde{\psi}_V}$  is the equilibrium/Gibbs state for the potential  $\tilde{\psi}_V$  and the shift map  $\sigma : E_V^{\mathbb{N}} \rightarrow E_V^{\mathbb{N}}$ .*

(d) *In addition,  $\tilde{\psi}_V$  is the amalgamated function of a summable Hölder continuous system of functions.*

*Proof.* Hölder continuity of  $\tilde{\psi}_V$  follows directly from (ESP) i.e Definition 18.5, and the fact that the function  $N_V$  is constant on cylinders of length one. Now, it follows from [6] that there exists a unique  $\exp(P(\psi) - \psi)$ -conformal measure on  $J(f)$ , i.e. a Borel probability measure  $m_\psi$  on  $J(f)$  such that

$$m_\psi(f(A)) = e^{P(\psi)} \int_A e^{-\psi} dm_\psi$$

for every Borel set  $A \subseteq J(f)$  such that the map  $f|_A$  is 1-to-1. In addition  $m_\psi$  is equivalent to  $\mu_\psi$  with logarithmically bounded Hölder continuous Radon-Nikodym derivative. It immediately follows from this formula that for every  $e \in E_V$  and every Borel set  $A \subseteq V$ , we have that

$$(18.4) \quad m_{\psi,V}(\varphi_e(A)) = \int_A \exp((\psi_V - P(\psi)\tau_V) \circ \varphi_e) dm_{\psi,V},$$

where  $m_{\psi,V}$  is the conditional measure of  $m_\psi$  on  $V$ . Now we define a Hölder continuous system of functions  $G := \{g^{(e)} : V \rightarrow \mathbb{R}\}_{e \in E}$  by putting

$$g^{(e)} := (\psi_V - P(\psi)\tau_V) \circ \varphi_e, \quad e \in E_V.$$

Formula (18.4) thus means that the system  $G$  is summable,  $P(G) = 0$ , and  $m_{\psi,V}$  is the unique  $G$ -conformal measure for the IFS  $\mathcal{S}_V$ . According to [14],  $g : E_V^{\mathbb{N}} \rightarrow \mathbb{R}$ , the amalgamated function of  $G$  is defined by the formula

$$\begin{aligned} g(\omega) &= g^{(\omega_1)}(\pi_V(\sigma(\omega))) = \psi_V \circ \varphi_{\omega_1}(\pi_V(\sigma(\omega))) - P(\psi)\tau_V \circ \varphi_{\omega_1}(\pi_V(\sigma(\omega))) \\ &= \psi_V \circ \pi_V(\omega) - P(\psi)N_V(\omega) \\ &= \tilde{\psi}_V(\omega). \end{aligned}$$

By Proposition 3.1.4 in [14] we thus have that

$$P(\sigma, \tilde{\psi}_V) = P(G) = 0.$$

Now, since  $\pi_V \circ \sigma = f_V \circ \pi_V$ , i.e. since the dynamical system  $f_V : J_V \rightarrow J_V$  is a factor of the shift map  $\sigma : E_V^{\mathbb{N}} \rightarrow E_V^{\mathbb{N}}$  via the map  $\pi_V : E_V^{\mathbb{N}} \rightarrow J_V$ , we see that  $\mu_{\tilde{\psi}_V} \circ \pi_V^{-1}$  is a Borel  $f_V$ -invariant probability measure on  $J_V$  equivalent to  $m_{\tilde{\psi}_V} \circ \pi_V^{-1} = m_g \circ \pi^{-1} = m_G = m_{\psi, V}$ . Since  $m_{\psi, V}$  is equivalent to  $\mu_{\psi, V}$ , we thus conclude that the measures  $m_{\tilde{\psi}_V} \circ \pi_V^{-1}$  and  $\mu_{\psi, V}$  are equivalent. Since both these measures are  $f_V$ -invariant and  $\mu_{\psi, V}$  is ergodic, they must be equal. The proof is thus complete.  $\square$

Since  $\pi_V : E_V^{\mathbb{N}} \rightarrow J_V = V_{\infty}$ , where, we recall the latter is the set of points returning infinitely often to  $V$ , is a measurable isomorphism sending the  $\sigma$ -invariant measure  $\mu_{\tilde{\psi}_V}$  to the  $f_V$ -invariant probability measure  $\mu_{\psi, V}$ , by identifying the sets  $E_V^{\mathbb{N}}$  and  $V_{\infty}(= J_V)$ , we can prove the following.

**Lemma 18.8.** *With the hypotheses of Proposition 18.7, the pentade  $(J(f), f, V, \tilde{\psi}_V, \mu_{\tilde{\psi}_V})$  is an SRT system and has exponential tail decay (ETD), where we recall that  $V_{\infty}$  is identified with  $E_V^{\mathbb{N}}$ ,  $\tilde{\psi}_V$  is identified with  $\psi_V - P(\psi)\tau_V$ , and  $\mu_{\tilde{\psi}_V}$  is identified with  $\mu_{\psi, V}$ .*

*Proof.* By virtue of Proposition 18.7 and Observation 16.2 we only need to prove that the pentade  $(J(f), f, V, \tilde{\psi}_V, \mu_{\tilde{\psi}_V})$  has exponential tail decay (ETD). We can assume without loss of generality that  $\psi : \hat{\mathbb{C}} \rightarrow \mathbb{R}$  is normalized so that

$$P(\psi) = 0 \quad \text{and} \quad m_{\psi} = \mu_{\psi}.$$

For every  $\geq 1$  denote by  $\mathcal{C}_V(n)$  the set of all connected components of  $T^{-n}(V)$ . Then define

$$\mathcal{C}_V^0(n) := \{U \in \mathcal{C}_V(n) : \forall_{(0 \leq k \leq n)} f^k(U) \cap V = \emptyset\}$$

Since the map  $f : J(f) \rightarrow J(f)$  is topologically exact, there exists an integer  $q \geq 1$  such that

$$f^q(V) \supseteq J(f).$$

Therefore for every  $e \in \mathcal{C}_V(n)$  there exists (at least one)  $\hat{e} \in \mathcal{C}_V^*(n+q)$  such that

$$f^q \circ \varphi_{\hat{e}} = \varphi_e.$$

By conformality of the measure  $\mu_{\psi}$ , for every  $e \in \mathcal{C}_V(n)$ , we have

$$\mu_{\psi}(\varphi_{\hat{e}}(V)) \geq \exp(-q\|\psi\|_{\infty})\mu_{\psi}(\varphi_e(V)).$$

So, since

$$\bigcup_{a \in \mathcal{C}_V^0(n+q)} \varphi_a(V) \subseteq \bigcup_{\substack{b \in \mathcal{C}_V(n+q) \\ f^q \circ \varphi_b \in \mathcal{C}_V^0(n)}} \varphi_b(V) \setminus \bigcup_{e \in \mathcal{C}_V^0(n)} \varphi_{\hat{e}}(V),$$

we therefore get

$$\begin{aligned}
\mu_\psi \left( \bigcup_{a \in \mathcal{C}_V^0(n+q)} \varphi_a(V) \right) &\leq \mu_\psi \left( \bigcup_{\substack{b \in \mathcal{C}_V(n+q) \\ f^q \circ \varphi_b \in \mathcal{C}_V^0(n)}} \varphi_b(V) \setminus \bigcup_{e \in \mathcal{C}_V^0(n)} \varphi_{\hat{e}}(V) \right) \\
&= \mu_\psi \left( \bigcup_{\substack{b \in \mathcal{C}_V(n+q) \\ f^q \circ \varphi_b \in \mathcal{C}_V^0(n)}} \varphi_b(V) \right) - \mu \left( \bigcup_{e \in \mathcal{C}_V^0(n)} \varphi_{\hat{e}}(V) \right) \\
&= \mu_\psi \left( f^{-q} \left( \bigcup_{c \in \mathcal{C}_V^0(n)} \varphi_c(V) \right) \right) - \sum_{e \in \mathcal{C}_V^0(n)} \mu_\psi(\varphi_{\hat{e}}(V)) \\
&= \mu_\psi \left( \bigcup_{c \in \mathcal{C}_V^0(n)} \varphi_c(V) \right) - \sum_{e \in \mathcal{C}_V^0(n)} \mu_\psi(\varphi_{\hat{e}}(V)) \\
&\leq \mu_\psi \left( \bigcup_{c \in \mathcal{C}_V^0(n)} \varphi_c(V) \right) - \exp(-q\|\psi\|_\infty) \sum_{e \in \mathcal{C}_V^0(n)} \mu_\psi(\varphi_e(V)) \\
&= \gamma \mu_\psi \left( \bigcup_{c \in \mathcal{C}_V^0(n)} \varphi_c(V) \right),
\end{aligned}$$

where  $\gamma := 1 - \exp(-q\|\psi\|_\infty) \in [0, 1)$ . An immediate induction then yields

$$\mu_\psi \left( \bigcup_{e \in \mathcal{C}_V^0(qn)} \varphi_e(V) \right) \leq \gamma^n$$

for all  $n \geq 0$ . An immediate induction then yields

$$\mu_\psi \left( \bigcup_{e \in \mathcal{C}_V^0(n)} \varphi_e(V) \right) \leq \gamma^{-1} \gamma^{n/q}$$

for all  $n \geq 0$ . But, as

$$E_V^{-1}([n, +\infty]) = E_V^{-1}(\{+\infty\}) \cup \bigcup_{k=n}^{\infty} \bigcup_{e \in \mathcal{C}_V^0(k)} \varphi_e(V)$$

and since  $\mu_\psi(E_V^{-1}(\{+\infty\})) = 0$  by ergodicity of  $\mu_\psi$  and of  $\mu_\psi(V) > 0$ , we therefore get that

$$(18.5) \quad \mu_\psi(E_V^{-1}([n, +\infty])) \leq (\gamma(1 - \gamma^{1/q}))^{-1} \gamma^{n/q}$$

for all  $n \geq 0$ . This just means that the pentade  $(I, f, V, \tilde{\psi}_V, \mu_{\tilde{\psi}_V})$  has exponential tail decay (ETD), and the proof is complete.  $\square$

Denote by  $J_R(f)$  the set of all recurrent points of  $f$  in  $J(f)$ . Formally

$$J_R(f) := \{z \in J(f) : \varliminf_{n \rightarrow \infty} |f^n(z) - z| = 0\}.$$

Of course  $J_R(f) \subseteq J_f$  and  $\mu_\psi(J(f) \setminus J_R(f)) = 0$  because of Poincaré's Recurrence Theorem. The set  $J_R(f)$  is significant for us since

$$J_R(f) \cap V \subseteq J_V.$$

Now we can now apply the conclusions of the work done. As a direct consequence of Theorem 10.10, Proposition 18.7, Lemma 18.8, Lemma 16.8, and Theorem 15.3, we get the following.

**Theorem 18.9.** *Let  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  be a tame rational function having the exponential shrinking property (ESP). Let  $\psi : \widehat{\mathbb{C}} \rightarrow \mathbb{R}$  be a Hölder continuous potential with pressure gap. Let  $z \in J_R(f) \setminus \overline{\text{PC}(f)}$ . Assume that the equilibrium state  $\mu_\psi$  is (WBT) at  $z$ . Then*

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{R_{\mu_\psi}(B(z, \varepsilon))}{\mu_\psi(B(z, \varepsilon))} &= \lim_{\varepsilon \rightarrow 0} \frac{\overline{R}_{\mu_\psi}(B(z, \varepsilon))}{\mu_\psi(B(z, \varepsilon))} \\ &= \begin{cases} 1 & \text{if } z \text{ is not any periodic point of } f, \\ 1 - \exp(S_p \psi(z) - pP(f, \psi)) & \text{if } z \text{ is a periodic point of } f. \end{cases} \end{aligned}$$

**Remark 18.10.** Theorem 18.9 holds in fact for a larger set than  $J_R(f)$ . Indeed, it holds for every point in  $V \cap J_{S_V}$ , where  $V$  is an arbitrary nice set.

As a fairly immediate consequence of Theorem 18.9 and Theorem 9.7, we get the following.

**Corollary 18.11.** *Let  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  be a tame rational function having the exponential shrinking property (ESP) whose Julia set  $J(f)$  is geometrically irreducible. If  $\psi : \widehat{\mathbb{C}} \rightarrow \mathbb{R}$  is a Hölder continuous potential with pressure gap, then*

$$\lim_{\varepsilon \rightarrow 0} \frac{R_{\mu_\psi}(B(z, \varepsilon))}{\mu_\psi(B(z, \varepsilon))} = \lim_{\varepsilon \rightarrow 0} \frac{\overline{R}_{\mu_\psi}(B(z, \varepsilon))}{\mu_\psi(B(z, \varepsilon))} = 1$$

for  $\mu_\psi$ -a.e.  $z \in J(f)$ .

Indeed in order to prove this corollary it suffices to note that if the Julia set  $J(f)$  is geometrically irreducible, then neither is the limit set of the iterated function system constructed in the arguments leading to Theorem 18.9.

**Remark 18.12.** We would like to note that if the rational function  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  is expanding, then it is tame, satisfies (ESP), and each Hölder continuous potential has pressure gap. In particular the two above theorems hold for it.

Now turn to the asymptotics of Hausdorff dimension. We recall the following.

**Definition 18.13.** Let  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  be a rational function of degree  $d \geq 2$ . We say that the map  $f$  is sub-expanding if one of the following two equivalent conditions holds:

(a)

$$\overline{\bigcup_{n=0}^{\infty} f^n(\text{Crit}(f) \setminus J(f)) \cap J(f)} = \emptyset \quad \text{and} \quad \overline{\text{Crit}(f) \cap \bigcup_{n=1}^{\infty} f^n(\text{Crit}(f) \cap J(f))} = \emptyset,$$

(b)

$$\overline{\text{Crit}(f) \cap \bigcup_{n=1}^{\infty} f^n(\text{Crit}(f) \cap J(f))} = \emptyset \quad \text{and} \quad f \text{ has no rationally indifferent periodic points.}$$

Let

$$h := \text{HD}(J(f)).$$

It was proved in [33] and [34] that there exists a unique  $h$ -conformal measure  $m_h$  on  $J(f)$  for  $f$  and a unique  $f$ -invariant (ergodic) measure  $\mu_h$  on  $J(f)$  equivalent to  $m_h$ . In addition  $\mu_h$  is supported on the intersection of the transitive and radial points of  $f$ . It has been proved in [34] that any subexpanding rational function enjoys ESP. It therefore follows from [20] that there are arbitrarily small open connected sets  $V_c$ ,  $c \in J(f) \cap \text{Crit}(f)$ , and  $V_\xi$ , respectively containing points  $c$  and  $\xi$  such that the collection of all holomorphic inverse branches  $f_*^{-n}$  of  $f^n$ ,  $n \geq 0$ , defined on  $V_z$ ,  $z \in (J(f) \cap \text{Crit}(f)) \cup \{\xi\}$ , and such that for some  $z' \in (J(f) \cap \text{Crit}(f)) \cup \{\xi\}$ ,

$$f_*^{-n}(V_z) \subseteq V_{z'}$$

and

$$\bigcup_{k=1}^{n-1} f^k(f_*^{-n}(V_z)) \cap \bigcup \{V_w : w \in (J(f) \cap \text{Crit}(f)) \cup \{\xi\}\} = \emptyset.$$

forms a finitely primitive conformal GDS, call it  $\mathcal{S}_f$ . Another characterization of  $\mathcal{S}_f$  is that its elements are composed of analytic inverse branches of the first return map of  $f$  from

$$V := \bigcup \{V_w : w \in (J(f) \cap \text{Crit}(f)) \cup \{\xi\}\}$$

$V$ . It has been proved in [27] and [27] that the system  $\mathcal{S}_f$  is strongly regular. It follows from Lemma 6.2 in [20] that  $\text{HD}(K(V)) < h$ . So, as by Theorem 12.1,  $\lim_{r \rightarrow 0} \text{HD}(K(B(\xi, r))) = h$ , we conclude that

$$\text{HD}(K(V)) < \text{HD}(K(B(\xi, r)))$$

for all  $r > 0$  small enough. Therefore, since  $h = b_{\mathcal{S}_f}$  and since  $\mu_{h,V} = \mu_{b_{\mathcal{S}_f}}$ , applying Theorem 12.1, Corollary 12.3, and Corollary 14.2, we get the following two theorems.

**Theorem 18.14.** *Let  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  be a subexpanding rational function of degree  $d \geq 2$ . Fix  $\xi \in J(f) \setminus \overline{\text{PC}(f)}$ . Assume that the measure  $\mu_h$  is (WBT) at  $\xi$  and the parameter  $h$  is*

powering at  $\xi$  with respect to the conformal GDS  $\mathcal{S}_f$ . Then the following limit exists, is finite and positive:

$$\lim_{r \rightarrow 0} \frac{\text{HD}(J(f)) - \text{HD}(K_\xi(r))}{\mu_h(B(\xi, r))}.$$

**Theorem 18.15.** *If  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  be a subexpanding rational function of degree  $d \geq 2$  whose Julia set  $J(f)$  is geometrically irreducible, then for  $\mu_h$ -a.e. point  $\xi \in J(f) \setminus \overline{\text{PC}(f)}$  the following limit exists, is finite and positive:*

$$\lim_{r \rightarrow 0} \frac{\text{HD}(J(f)) - \text{HD}(K_\xi(r))}{\mu_h(B(\xi, r))}.$$

**Remark 18.16.** We would like to note that if the rational function  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  is expanding, then it is automatically subexpanding and the two above theorems apply.

## 19. ESCAPE RATES FOR MEROMORPHIC FUNCTIONS ON THE COMPLEX PLANE

We deal in this final section with transcendental meromorphic functions. We also apply here the results on escape rates for conformal GDMS and the techniques of first return maps. Let  $f : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$  be a meromorphic function. Let  $\text{Sing}(f^{-1})$  be the set of all singular points of  $f^{-1}$ , i. e. the set of all points  $w \in \widehat{\mathbb{C}}$  such that if  $W$  is any open connected neighborhood of  $w$ , then there exists a connected component  $U$  of  $f^{-1}(W)$  such that the map  $f : U \rightarrow W$  is not bijective. Of course if  $f$  is a rational function, then  $\text{Sing}(f^{-1}) = f(\text{Crit}(f))$ . As in the case of rational functions, we define

$$\text{PS}(f) := \bigcup_{n=0}^{\infty} f^n(\text{Sing}(f^{-1})).$$

The function  $f$  is called *topologically hyperbolic* if

$$\text{dist}_{\text{Euclid}}(J_f, \text{PS}(f)) > 0,$$

and it is called *expanding* if there exist  $c > 0$  and  $\lambda > 1$  such that

$$|(f^n)'(z)| \geq c\lambda^n$$

for all integers  $n \geq 1$  and all points  $z \in J_f \setminus f^{-n}(\infty)$ . Note that every topologically hyperbolic meromorphic function is tame. A meromorphic function that is both topologically hyperbolic and expanding is called *hyperbolic*. The meromorphic function  $f : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$  is called dynamically *semi-regular* if it is of finite order, commonly denoted by  $\rho_f$ , and satisfies the following rapid growth condition for its derivative.

$$(19.1) \quad |f'(z)| \geq \kappa^{-1}(1 + |z|)^{\alpha_1}(1 + |f(z)|)^{\alpha_2}, \quad z \in J_f,$$

with some constant  $\kappa > 0$  and  $\alpha_1, \alpha_2$  such that  $\alpha_2 > \max\{-\alpha_1, 0\}$ . Set  $\alpha := \alpha_1 + \alpha_2$ .

**Remark 19.1.** A particularly simple example of such maps are meromorphic functions  $f_\lambda(z) = \lambda e^z$  where  $\lambda \in (0, 1/e)$  since these maps have an attracting periodic point. A good reference is [16].

Let  $h : J_f \rightarrow \mathbb{R}$  be a weakly Hölder continuous function in the sense of [17]. The definition, introduced in [17] is somewhat technical and we will not provide it in the current paper. What is important is that each bounded, uniformly locally Hölder function  $h : J_f \rightarrow \mathbb{R}$  is weakly Hölder. Fix  $\tau > \alpha_2$  as required in [17]. For  $t \in \mathbb{R}$ , let

$$(19.2) \quad \psi_{t,h} = -t \log |f'|_\tau + h$$

where  $|f'(z)|_\tau$  is the norm, or, equivalently, the scaling factor, of the derivative of  $f$  evaluated at a point  $z \in J_f$  with respect to the Riemannian metric

$$|d\tau(z)| = (1 + |z|)^{-\tau} |dz|.$$

Following [17] functions of the form (19.2) (frequently referred to as potentials) are called *loosely tame*. Let  $\mathcal{L}_{t,h} : C_b(J_f) \rightarrow C_b(J_f)$  be the corresponding *Perron-Frobenius operator* given by the formula

$$\mathcal{L}_{t,h}g(z) := \sum_{w \in f^{-1}(z)} g(w) e^{\psi_{t,h}(w)}.$$

It was shown in [17] that, for every  $z \in J_f$  and for the function  $\mathbb{1} : z \mapsto 1$ , the limit

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{L}_{t,h}^n \mathbb{1}(z)$$

exists and takes on the same common value, which we denote by  $P(t)$  and call *the topological pressure* of the potential  $\psi_t$ . The following theorem was proved in [17].

**Theorem 19.2.** *If  $f : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$  is a dynamically semi-regular meromorphic function and  $h : J_f \rightarrow \mathbb{R}$  is a weakly Hölder continuous potential, then for every  $t > \rho_f/\alpha$  there exist uniquely determined Borel probability measures  $m_{t,h}$  and  $\mu_{t,h}$  on  $J_f$  with the following properties.*

- (a)  $\mathcal{L}_{t,h}^* m_{t,h} = m_{t,h}$ .
- (b)  $P(\psi_{t,h}) = \sup \left\{ h_\mu(f) + \int \psi_{t,h} d\mu : \mu \circ f^{-1} = \mu \text{ and } \int \psi_{t,h} d\mu > -\infty \right\}$ .
- (c)  $\mu_{t,h} \circ f^{-1} = \mu_{t,h}$ ,  $\int \psi_{t,h} d\mu_{t,h} > -\infty$ , and  $h_{\mu_{t,h}}(f) + \int \psi_{t,h} d\mu_{t,h} = P(\psi_{t,h})$ .
- (d) *The measures  $\mu_{t,h}$  and  $m_{t,h}$  are equivalent and the Radon-Nikodym derivative  $\frac{d\mu_{t,h}}{dm_{t,h}}$  has a nowhere-vanishing Hölder continuous version which is bounded above.*

The exact analogue of Theorem 18.4 holds, with the same references, for all hyperbolic meromorphic functions; we will refer to this theorem as Theorem 18.4(M). Also, for the system  $\mathcal{S}_V$  and the projection  $\pi_V : E_V^{\mathbb{N}} \rightarrow J_V$  have the same meaning. As in the case of rational functions denote by  $J_R(f)$  the set of all recurrent points of  $f$  in  $J(f)$ . Formally

$$J_R(f) := \{z \in J(f) : \varliminf_{n \rightarrow \infty} |f^n(z) - z| = 0\}.$$

Of course  $J_R(f) \subseteq J_f$  and  $\mu_\psi(J(f) \setminus J_R(f)) = 0$  because of Poincaré's Recurrence Theorem. The set  $J_R(f)$  is significant for us since

$$J_R(f) \cap V \subseteq J_V.$$

The Exponential Shrinking Property (ESP) holds since now the function  $f : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$  is expanding. The proof of Proposition 18.7 goes through unchanged except that instead of using [6] we now invoke Theorem 19.2 (a). We also will refer this proposition (18.7) as Proposition 18.7 (M). Lemma 18.8 also carries on to the meromorphic case (we refer to it as Lemma 18.8 (M); the proof of items (a)–(e) Definition 16.1 required by this lemma to hold, follows as in the case of rational functions, from proposition 18.7 (M), while the proof of item (f) of this definition is now a direct consequence of Lemma 4.1 in [26]. Now, in exactly the same way as in the case of rational functions, as a direct consequence of Theorem 10.10, Theorem 10.11, Proposition 18.7 (M), Lemma 18.8 (M), Lemma 16.8, and Theorem 15.3, we get the following two theorems.

**Theorem 19.3.** *Let  $f : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$  be a dynamically semi-regular meromorphic function. Let  $t > \rho_f/\alpha$  and let  $h : J(f) \rightarrow \mathbb{R}$  be a weakly Hölder continuous function. Let  $z \in J_R(f)$ . Assume that the corresponding equilibrium state  $\mu_{t,h}$  is (WBT) at  $z$ . Then*

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{R_{\mu_{t,h}}(B(z, \varepsilon))}{\mu_{t,h}(B(z, \varepsilon))} &= \lim_{\varepsilon \rightarrow 0} \frac{\overline{R}_{\mu_{t,h}}(B(z, \varepsilon))}{\mu_{t,h}(B(z, \varepsilon))} = \\ &= \begin{cases} 1 & \text{if } z \text{ is not any periodic point of } f, \\ 1 - \exp(S_p \psi_{t,h}(z) - pP(\psi_{t,h})) & \text{if } z \text{ is a periodic point of } f. \end{cases} \end{aligned}$$

**Theorem 19.4.** *Let  $f : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$  be a dynamically semi-regular meromorphic function whose Julia set is geometrically irreducible. Let  $t > \rho_f/\alpha$  and let  $h : J(f) \rightarrow \mathbb{R}$  be a weakly Hölder continuous function. Then*

$$\lim_{\varepsilon \rightarrow 0} \frac{R_{\mu_{t,h}}(B(z, \varepsilon))}{\mu_{t,h}(B(z, \varepsilon))} = \lim_{\varepsilon \rightarrow 0} \frac{\overline{R}_{\mu_{t,h}}(B(z, \varepsilon))}{\mu_{t,h}(B(z, \varepsilon))} = 1$$

for  $\mu_{t,h}$ -a.e.  $z \in J(f)$ .

**Remark 19.5.** Theorem 19.3 holds in fact for a larger set than  $J_R(f)$ . Indeed, it holds for every point in  $V \cap J_{S_V}$ , where  $V$  is an arbitrary nice set.

Turning to the asymptotics of Hausdorff dimension, let  $J_r(f)$  be the set of radial (or conical) points in  $J(f)$ , i. e. the set of all those points in  $J(f)$  that do not escape to infinity under the action of the map  $f : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$ . Assume now more, namely that  $f : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$  is dynamically regular in the sense of [16] and [17]. What at the moment is important for us is that  $P(h_r) = 0$ , where

$$h_r := \text{HD}(J_r(f)).$$

We already know that there exists a nice set  $V$  containing  $\xi$  and the elements of the corresponding conformal IFS  $\mathcal{S}_f$  are composed of analytic inverse branches of the first return map from  $V$  to  $V$ . Since  $\xi \in J_R(f)$ , we have that  $\xi \in J_V$ . Corollary 6.4 in [25]



tells us that  $\text{HD}(K(V)) < h_r$ . So, since by Theorem 12.1,  $\lim_{r \rightarrow 0} \text{HD}(K(B(\xi, r))) = h_r$ , we conclude that

$$\text{HD}(K(V)) < \text{HD}(K(B(\xi, r)))$$

for all  $r > 0$  small enough. Therefore, since  $h_r = b_{S_V}$  and since  $\mu_{h,V} = \mu_{b_{S_V}}$ , applying Theorem 12.1, Corollary 12.3, and Corollary 14.2, we get the following two theorems.

**Theorem 19.6.** *Let  $f : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$  be a dynamically regular meromorphic function. Fix  $\xi \in J_R(f)$ . Assume that the measure  $\mu_{h_r}$  (i.e.  $\mu_{h_r,0}$  with the weakly Hölder function  $h$  identically equal to 0) is (WBT) at  $\xi$  and the parameter  $h_r$  is powering at  $\xi$  with respect to the conformal IFS  $\mathcal{S}_f$ . Then the following limit exists and is finite and positive:*

$$\lim_{r \rightarrow 0} \frac{\text{HD}(J_r(f)) - \text{HD}(K_z(r))}{\mu_{h_r}(B(z, r))}.$$

**Theorem 19.7.** *Let  $f : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$  be a dynamically regular meromorphic function whose Julia set is geometrically irreducible. Then the following limit exists and is finite and positive for  $\mu_{h_r}$ -a.e.  $z \in J(f)$ :*

$$\lim_{r \rightarrow 0} \frac{\text{HD}(J_r(f)) - \text{HD}(K_z(r))}{\mu_{h_r}(B(z, r))}.$$

Note that the conclusion of Remark 19.5 holds in the case of Theorem 19.6 too.

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UNIVERSITY OF WARWICK, INSTITUTE OF MTHEMATICS, UK

*E-mail address:* `masdbl@warwick.ac.uk`

UNIVERSITY OF NORTH TEXAS, DEPARTMENT OF MATHEMATICS, 1155 UNION CIRCLE #311430,  
DENTON, TX 76203-5017, USA

*E-mail address:* `urbanski@unt.edu`

**Web:** `www.math.unt.edu/~urbanski`